Constructing the Bijective BWT and the Extended Burrows-Wheeler Transform in Linear Time

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[CPM ‘21]

the bijective BWT (BBWT) is
the BWT of
the Lyndon factorization
of an input text
with respect to $\prec_\omega$

the bijective BWT (BBWT) is
the BWT of
the Lyndon factorization 1.
of an input text
with respect to $\prec_\omega$ 2.

Lyndon words

- `a`
- `aabab`

Lyndon word is smaller than
- any proper suffix
- any rotation

Lyndon words

- a
- aabab

Lyndon word is smaller than
- any proper suffix
- any rotation

not Lyndon words:

- abaab (rotation aabab smaller)
- abab (abab not smaller than suffix ab)

Lyndon factorization [Chen+ '58]

- input: text T = 

- output: factorization $T_1 ... T_t$ with
  - T_x is Lyndon word
  - $T_x \geqslant_{\text{lex}} T_{x+1}$
  - factorization uniquely defined
  - linear time [Duval '88]

(Chen-Fox-Lyndon Theorem)

example

T = senescence

Lyndon factorization: s|enes|cen|ce

- s, enes, cen, and ce are Lyndon
- $s >_{lex} enes >_{lex} cen >_{lex} ce$

$\prec_\omega$ order

- $u \prec_\omega w :\Longleftrightarrow uuuu... <_{\text{lex}} wwww...$

- $\mathtt{ab} <_{\text{lex}} \mathtt{aba}$
- $\mathtt{aba} \prec_\omega \mathtt{ab}$

$\prec_\omega$ order

- $u \prec_\omega w :\Longleftrightarrow uuuu\ldots <_{\text{lex}} wwww\ldots$

- $ab <_{\text{lex}} aba$
- $aba \prec_\omega ab$

abababab⋯
ababaaba⋯

conjugates

- $T = T[1]\,T[2]\,\cdots\,T[n]$
- conjugates = cyclic shifts:
  - $T[1]\,T[2]\,\cdots\,T[n]$
  - $T[2]\,T[3]\,\cdots\,T[n]\,T[1]$
  - $\vdots$
  - $T[n]\,T[1]\,\cdots\,T[n\text{-}1]$

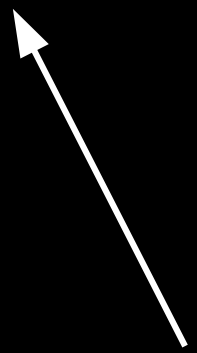
bijective BWT of senescence

s|enes|cen|ce

bijective BWT of senescence

s|enes|cen|ce

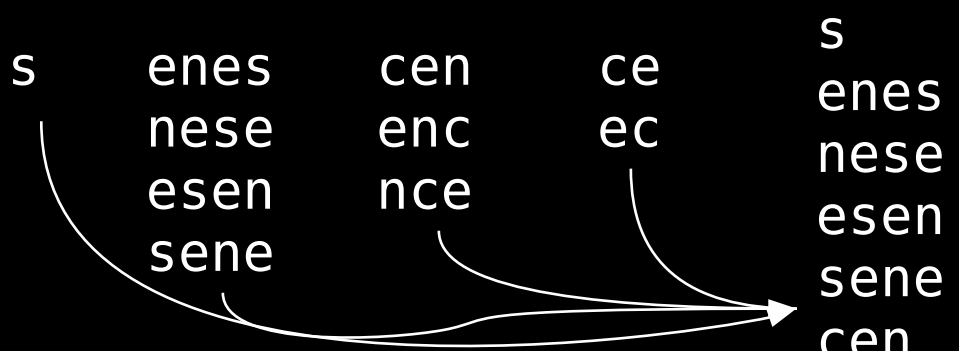
```
s    enes    cen    ce
     nese    enc    ec
     esen    nce
     sene
```



conjugates of all Lyndon factors

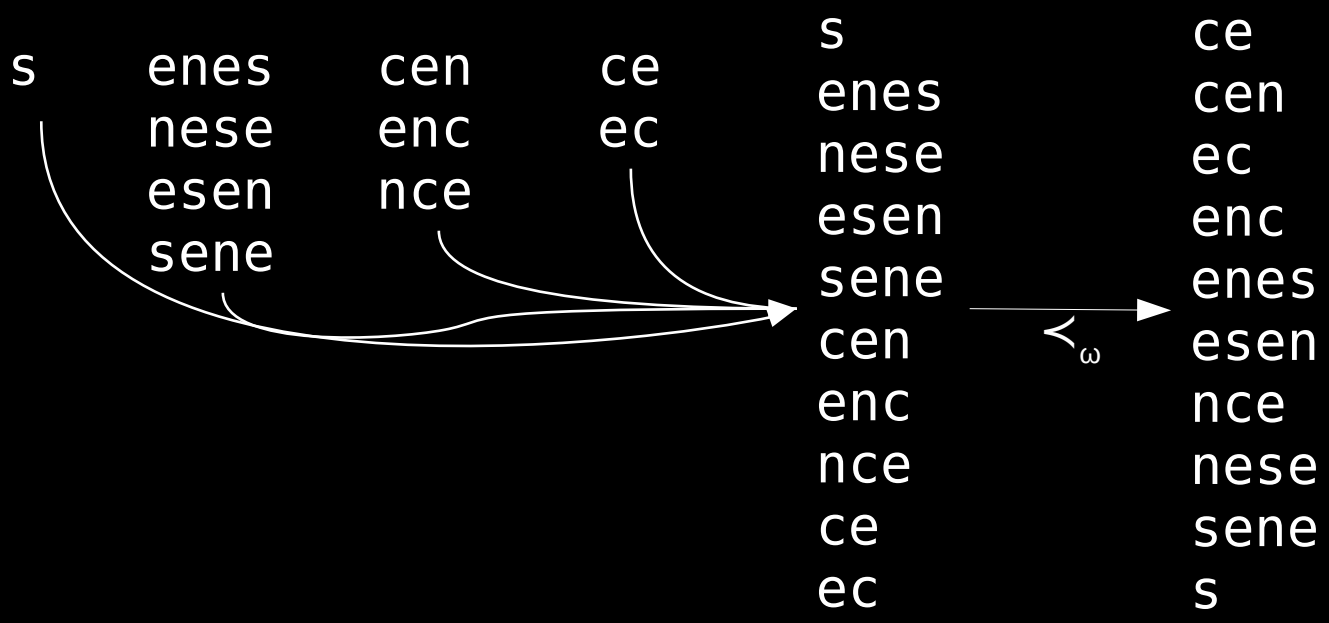
bijective BWT of senescence

s|enes|cen|ce



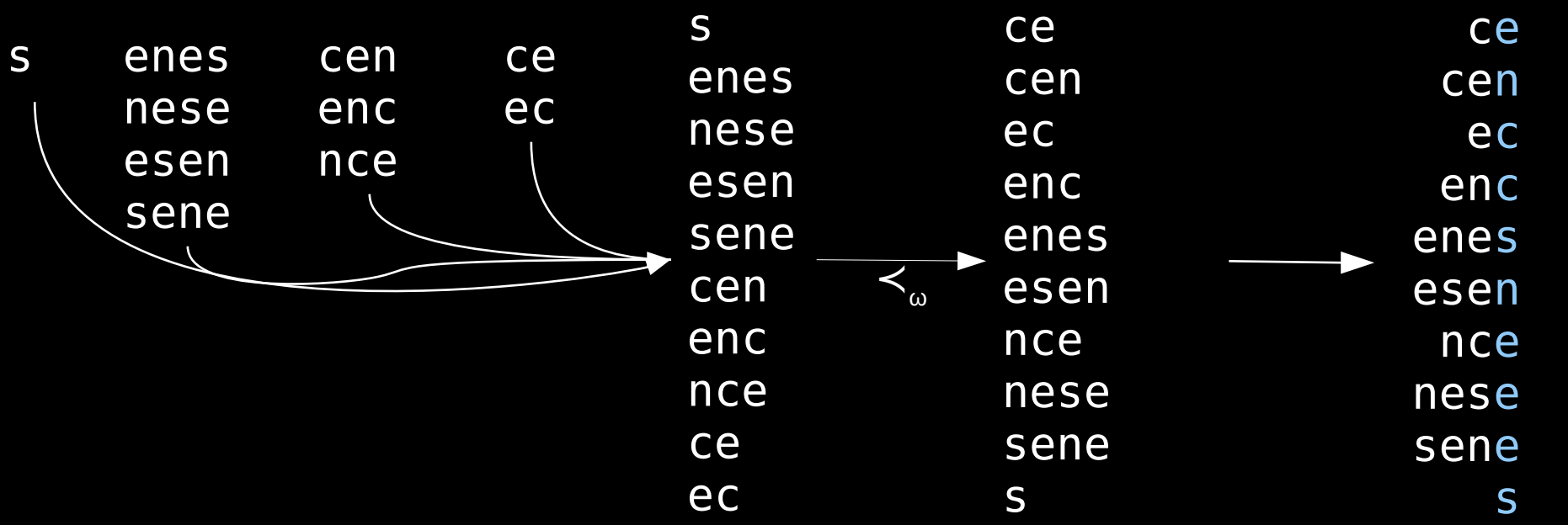
bijective BWT of senescence

s|enes|cen|ce



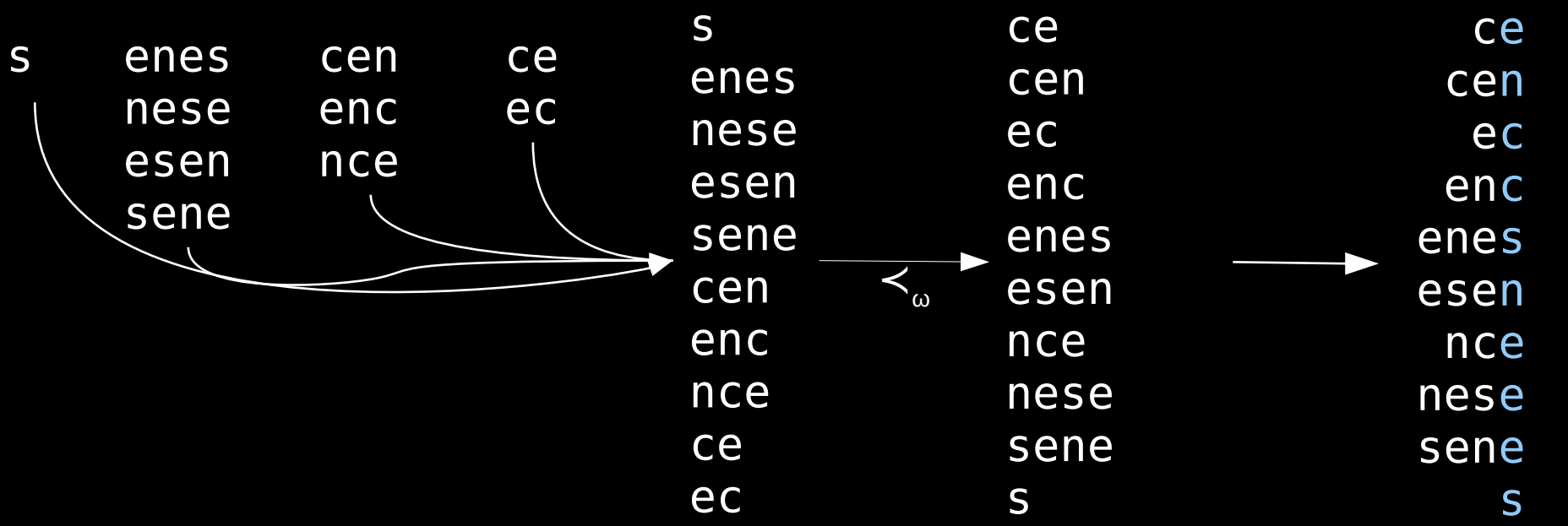
bijective BWT of senescence

s|enes|cen|ce



bijective BWT of senescence

s|enes|cen|ce



result: enccsneees

motivation

properties of BBWT :

- no $ necessary
- BBWT seems to be more compressible than BWT for some inputs

[Scott and Gill '12]

- extended BWT : special case of BBWT

[Mantaci+ ‘07]

- BBWT is indexible (full text index)

[Bannai+ '19]

however, linear time construction was only conjectured!

time

- how much time does it take to sort?
- #conjugates = number of text positions = n

  $\Rightarrow$ naively: $O(n^2)$ time

- can we use $O(n)$ time suffix sorting?

BWT of senescence

senescence

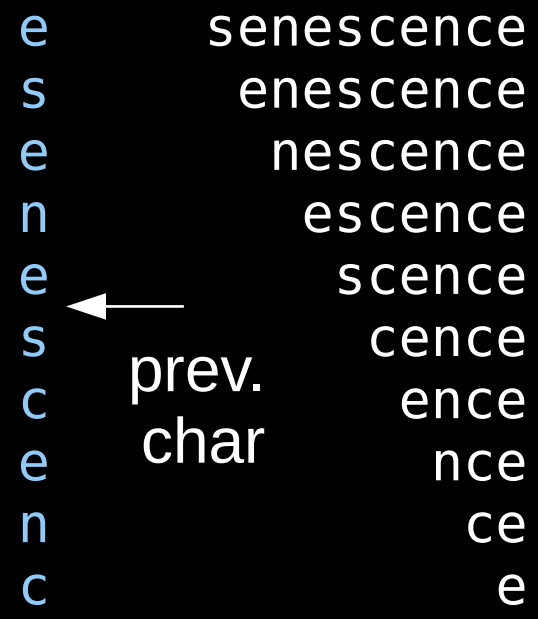
BWT of senescence

senescence

```
senescence
 enescence
  nescence
   escence
    scence
     cence
      ence
       nce
        ce
         e
```

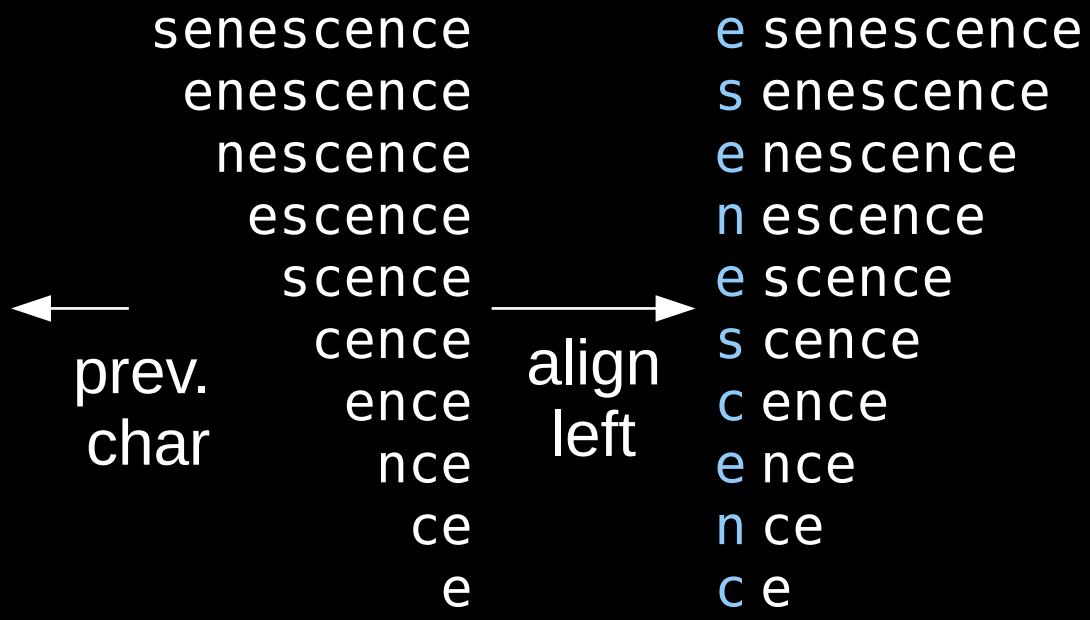
BWT of senescence

senescence

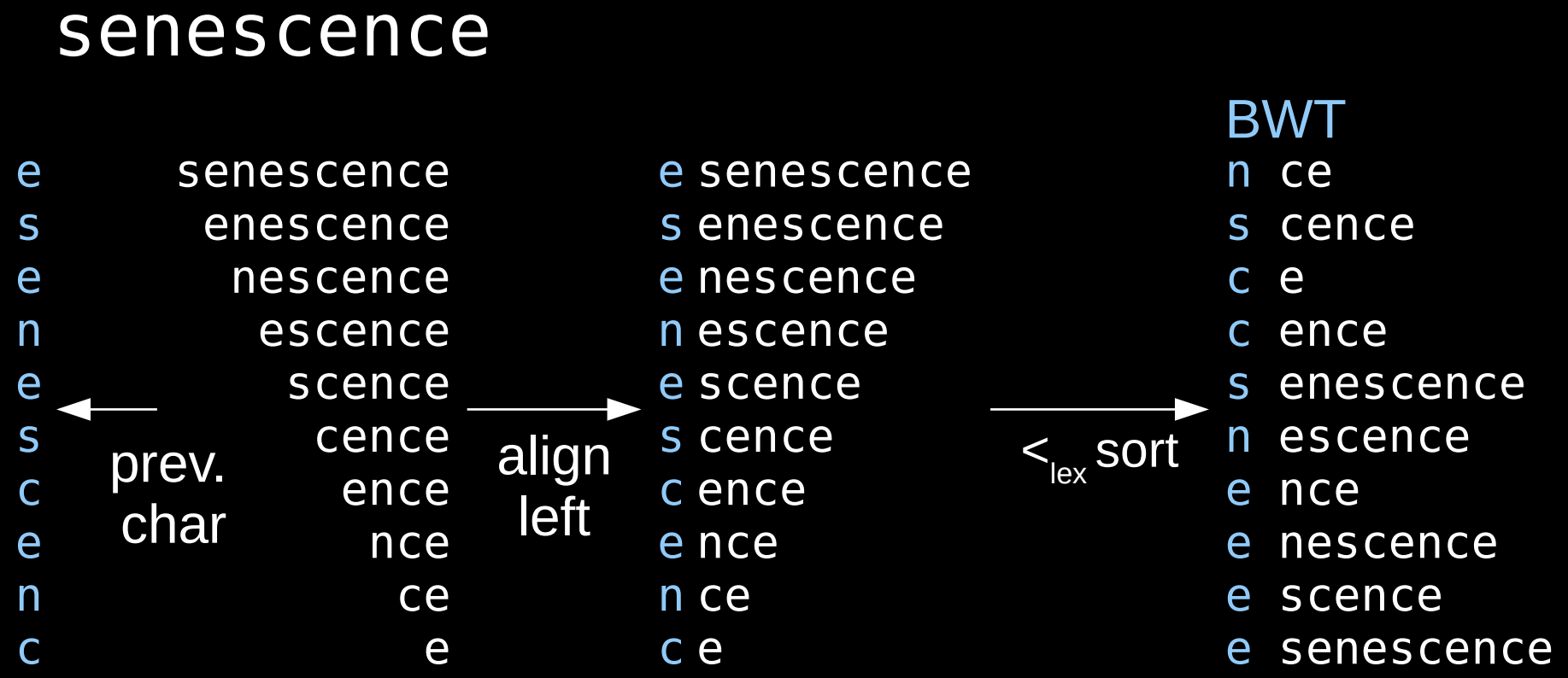


BWT of senescence

senescence

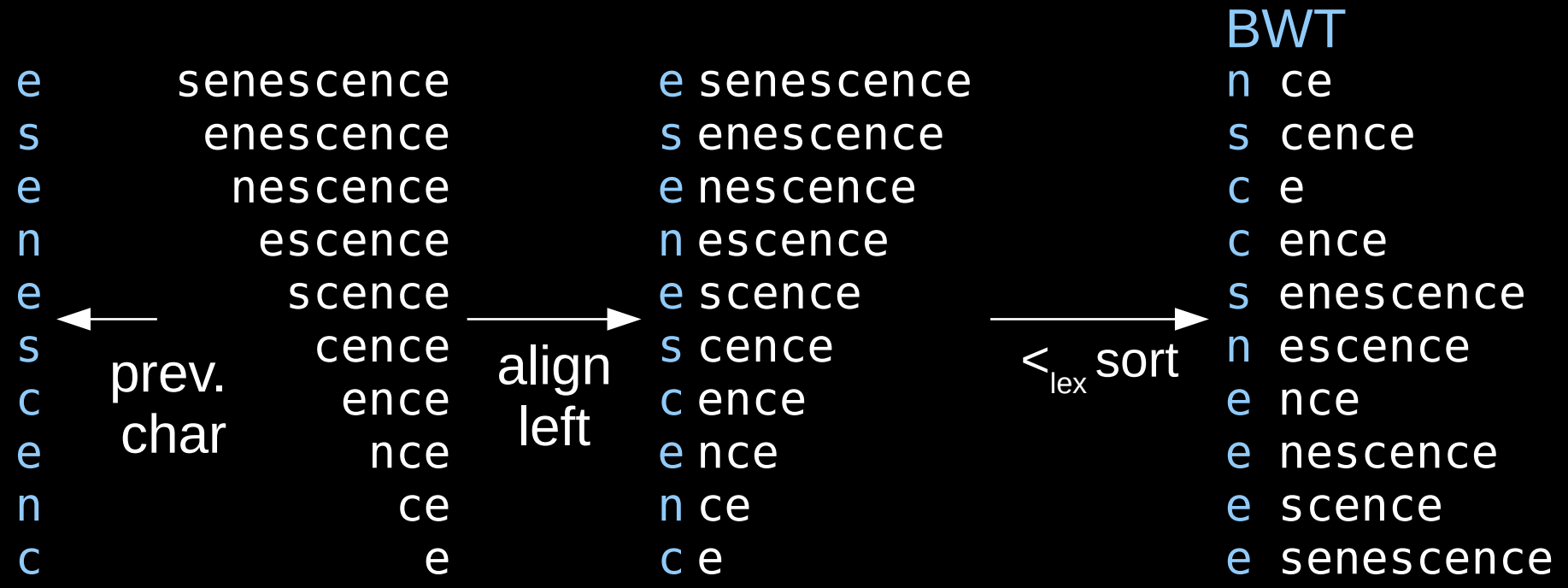


BWT of senescence



BWT of senescence

senescence



output: nsccsneeee (BBWT: enccsneees)

BBWT of senescence

s|enes|cen|ce

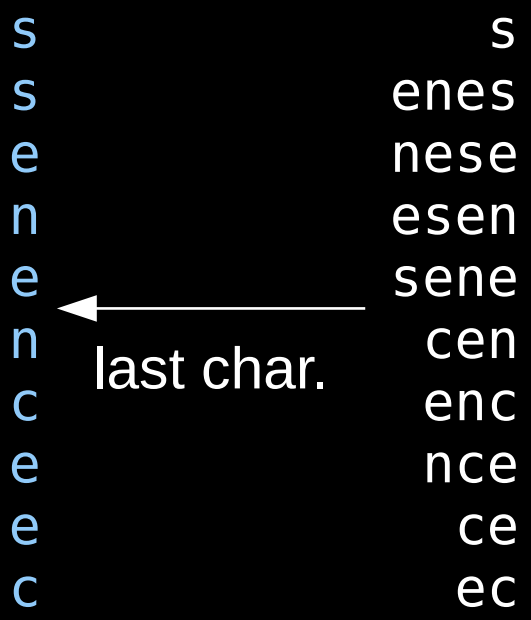
BBWT of senescence

s|enes|cen|ce

```
   s
enes
nese
esen
sene
 cen
 enc
 nce
  ce
  ec
```

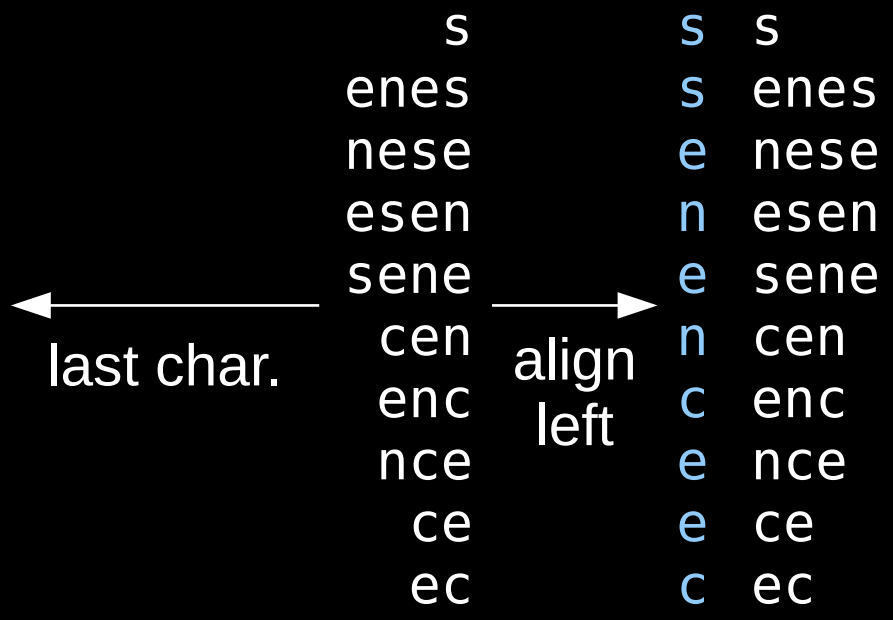
BBWT of senescence

s|enes|cen|ce



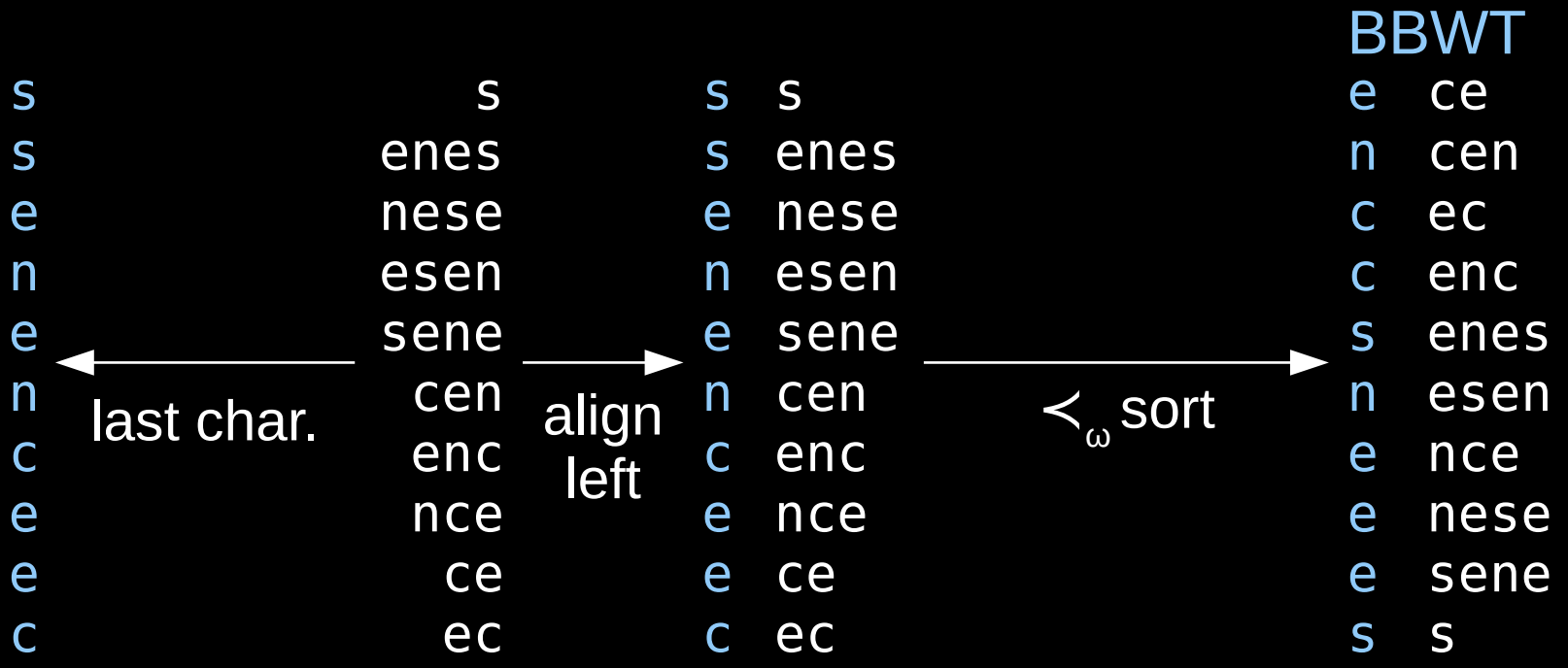
BBWT of senescence

s|enes|cen|ce



BBWT of senescence

s|enes|cen|ce



BBWT of senescence

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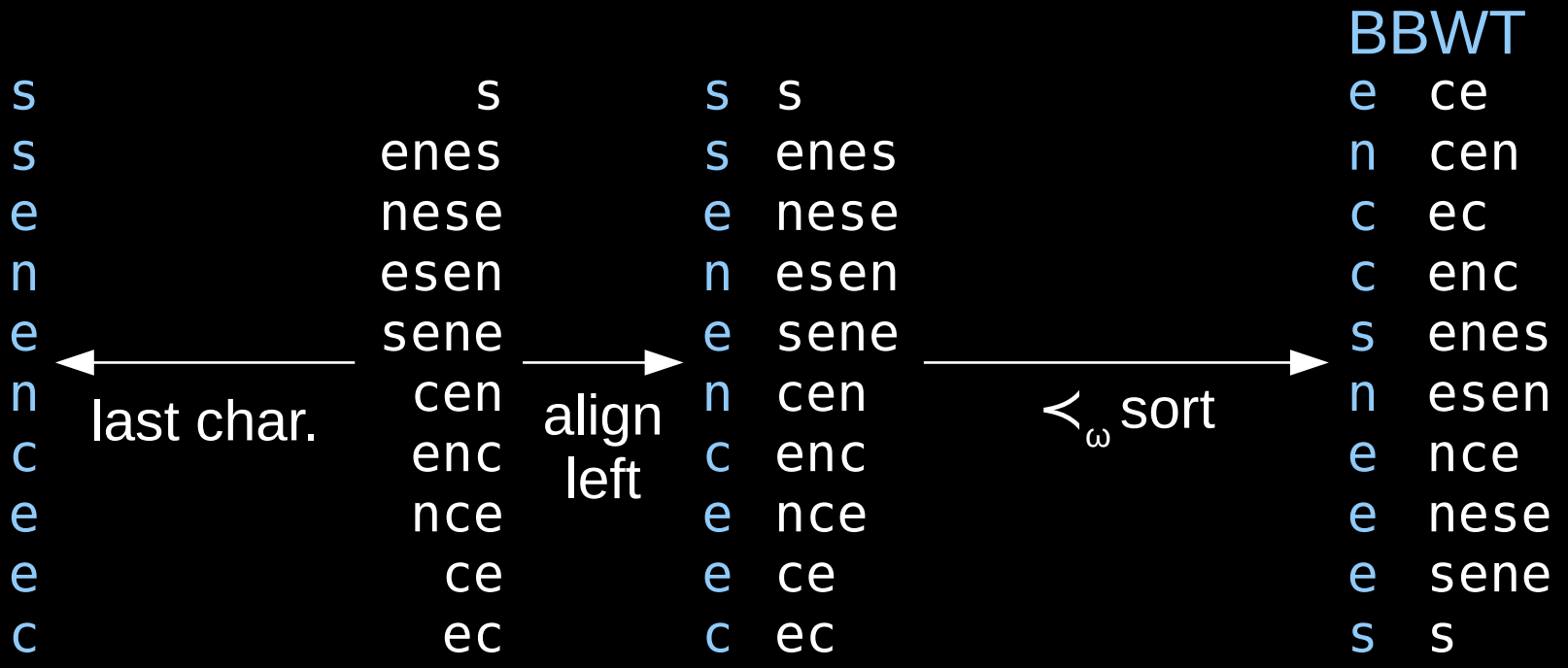


output: enccsneees

SA ⇒ BWT

- BWT[*i*] = *T* [SA[*i*]-1]
- SA[*i*] = starting position of the *i* th smallest suffix respective to lexicographic order

SAIS:

- well-known SA construction algorithm
- O(*n*) time for integer alphabets

[Nong+ ’11]

LSA ⇒ BBWT

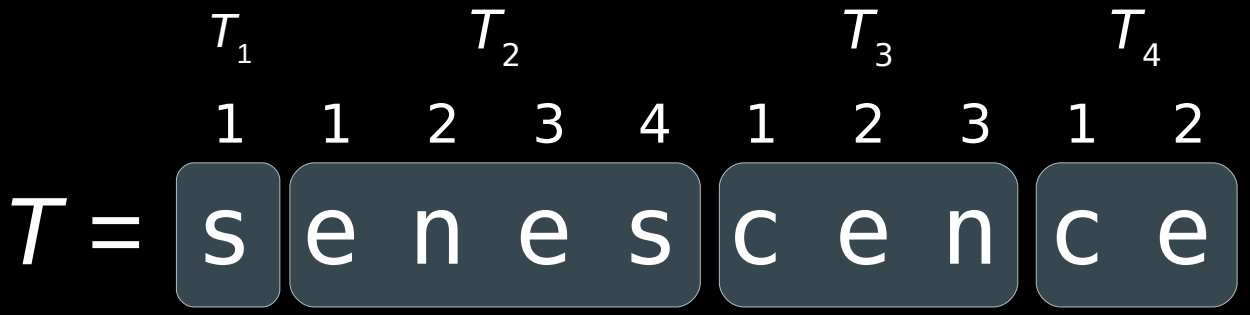
- BBWT[*i*] = *T* [LSA[*i*]-1]
- LSA[*i*] = starting position of *i* th smallest conjucate respective to $\prec_\omega$ order

[Hon+ '12]

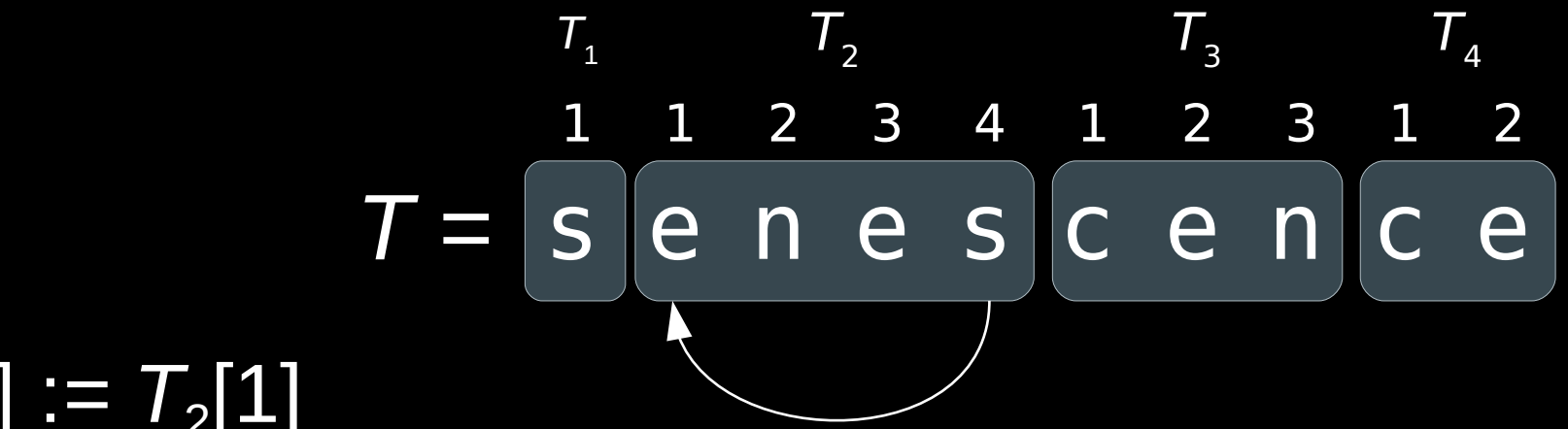
LSAIS: LSA construction algorithm

- sorts conjugates instead of suffixes
- uses $\prec_\omega$ order instead of lex. order

rewinding

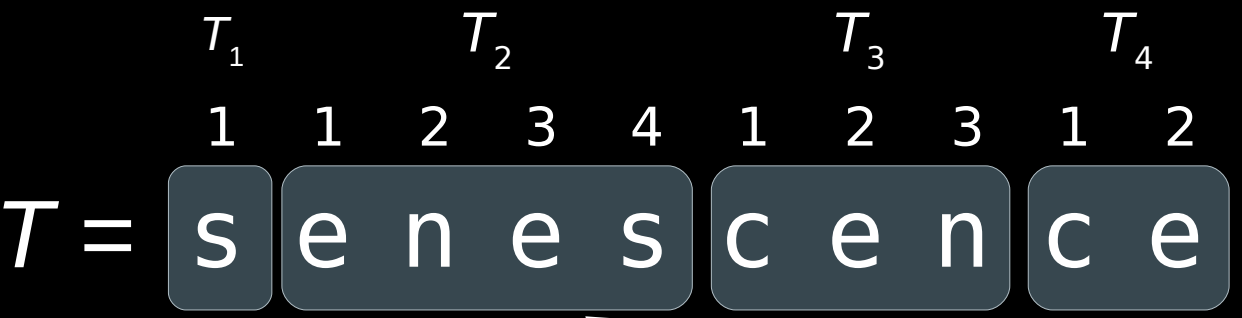


rewinding



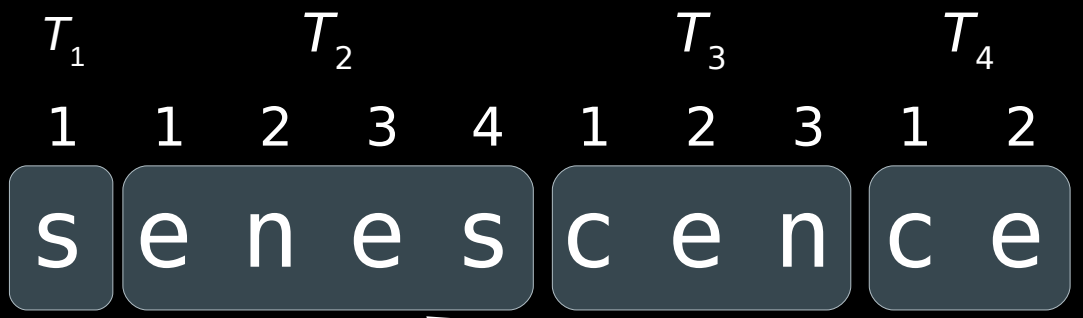
- $T_2[5] := T_2[1]$

rewinding



- $T_2[5] := T_2[1]$
- $T_2[4..] := T_2[4]T_2[1]T_2[2]T_2[3]T_2[4]T_2[1]...$

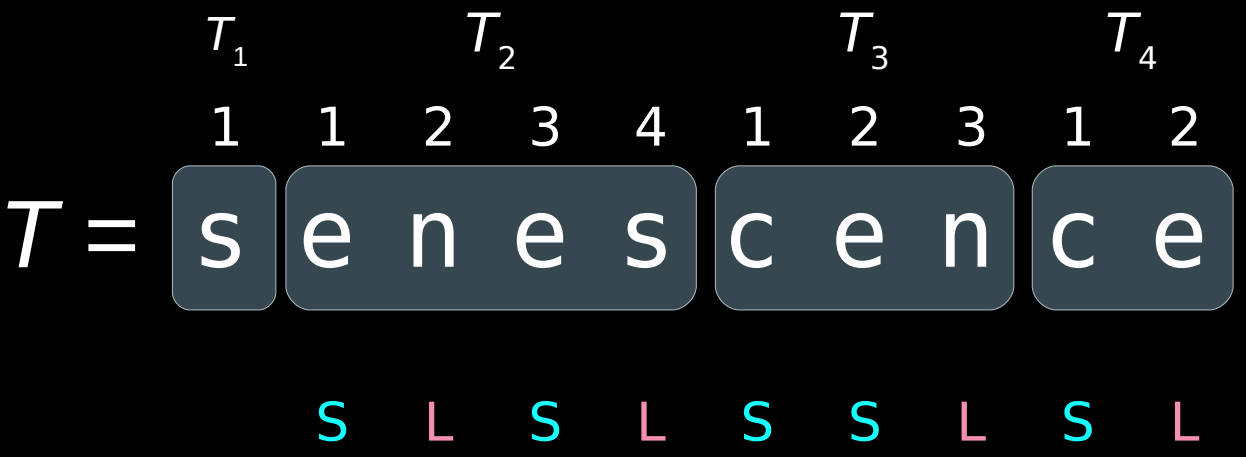
rewinding



- $T_2[5] := T_2[1]$
- $T_2[4..] := T_2[4]T_2[1]T_2[2]T_2[3]T_2[4]T_2[1]...$

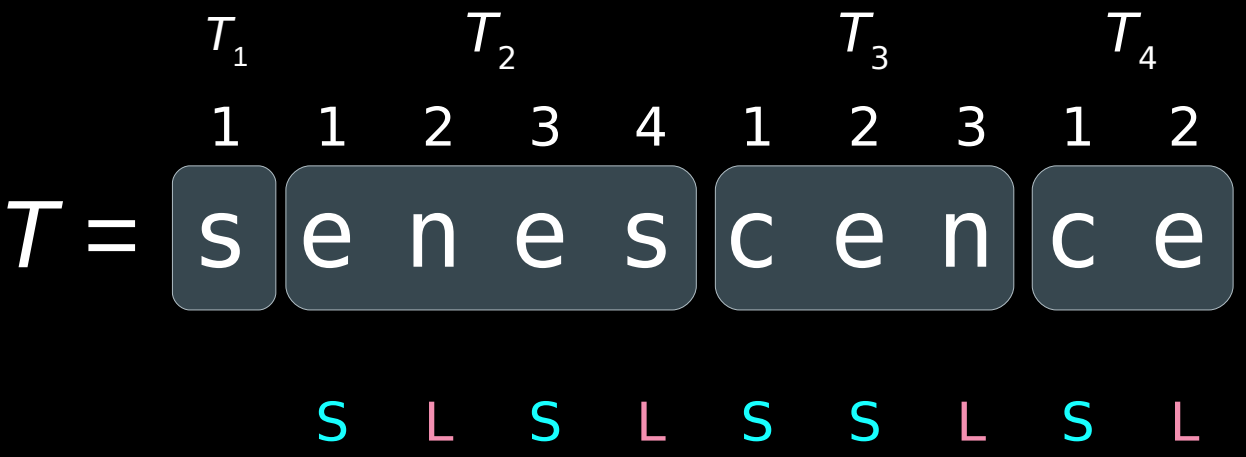
  in general, for any x define:
  - $T_x[|T_x|+j\ ] := T_x[(|T_x|+j\text{-}1) \bmod |T_x|+1]\ ,\ \ j \geq 0$
  - $T_x[0] := T_x[|T_x|]$
  - $T_x[\ i..] := T_x[\ i\ ]\ ...\ T_x[|T_x|]\ T_x[1]\ ...$

L / S type



- $T_x[\,i\,] <_{lex} T_x[\,i{+}1] \Rightarrow T_x[\,i\,]$ is S type
- $T_x[\,i\,] >_{lex} T_x[\,i{+}1] \Rightarrow T_x[\,i\,]$ is L type
- $T_x[\,i\,] = T_x[\,i{+}1] \Rightarrow T_x[\,i\,]$ and $T_x[\,i{+}1]$ are same type

L / S type



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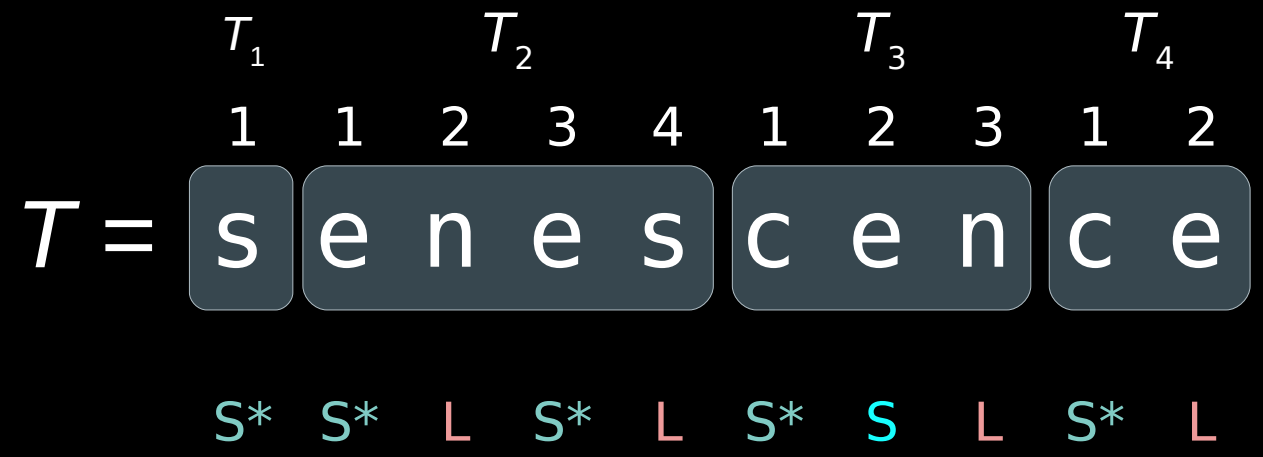
- $T_x[\,i\,] = T_x[\,i{+}1] \Rightarrow T_x[\,i\,]$ and $T_x[\,i{+}1]$ are same type

thanks to the Lyndon factorization the S / L types are the same as in the original SAIS

S* type



- if $T_x[\ i\ ]$ is S type and $T_x[\ i\text{-}1\ ]$ is L type
  ⇒ $T_x[\ i\ ]$ is S* type
- $T_x[1]$ is always S* type ∀ x

LMS (substrings)

	1	2	3	4	5	6	7	8	9	10
T =	s	e	n	e	s	c	e	n	c	e
	S*	S*	L	S*	L	S*	S	L	S*	L

for $1 \le i < j \le |T_x|+1$:

$T_x[\, i..j \,]$ is called LMS if

- $T_x[\, i \,]$ and $T_x[\, j \,]$ are S* type and
- $T_x[\, k \,]$ is not S* type $\forall\, k \in [i+1 \,..\, j-1]$

start pos.	LMS
1	ss
2	ene
4	ese
6	cenc
9	cec

LSAIS algorithm

1) sort all LMS in $<_{lex}$
2) place S* types in LSA
3) induce L types
4) induce S types

sort all LMS

start pos.	LMS
1	ss
2	ene
4	ese
6	cenc
9	cec

sort all LMS

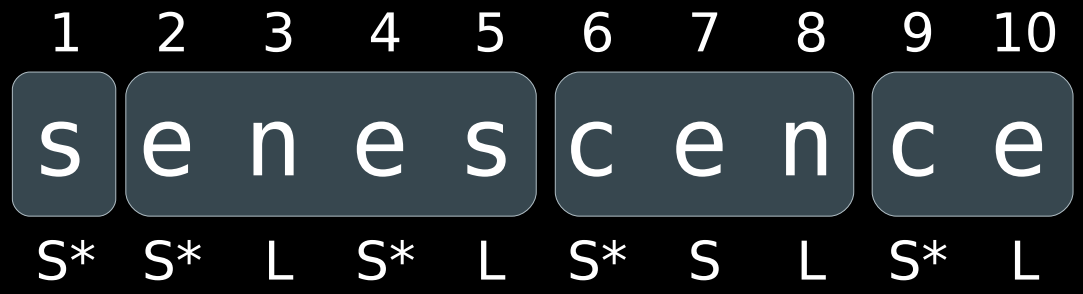
start pos.	LMS
1	ss
2	ene
4	ese
6	cenc
9	cec

$<_{lex}$ sort

start pos.	LMS
9	cec
6	cenc
2	ene
4	ese
1	ss

works by recursion like SAIS,
see paper for details

S/L type bucket allocation



LSA =

S		L	S			L		L	S
c		e				n		s	

place S* types

pos.	LMS
9	cec
6	cenc
2	ene
4	ese
1	ss

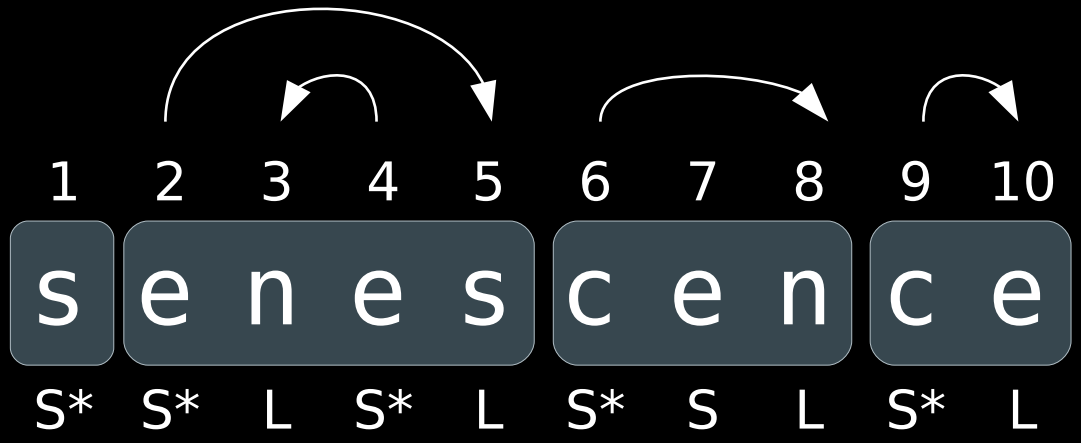
T =

1	2	3	4	5	6	7	8	9	10
s	e	n	e	s	c	e	n	c	e
S*	S*	L	S*	L	S*	S	L	S*	L

LSA =

9	6		2	4					1
S		L	S			L		L	S
c			e			n		s	

induce L types



LSA =

9	6	10	2	4		8	3	5	1
S		L	S			L		L	S
c		e				n		s	

induce S types

LSA =

9	6	10	7	2	4	8	3	5	1
S		L	S			L		L	S
c		e				n		s	

BBWT[*i*] = *T* [LSA[*i*]-1]

	1	2	3	4	5	6	7	8	9	10
T =	s	e	n	e	s	c	e	n	c	e
	S*	S*	L	S*	L	S*	S	L	S*	L

LSA =	9	6	10	7	2	4	8	3	5	1

LSA-1 =	10	8	9	6	5	3	7	2	4	1
T [*LSA*-1] =	e	n	c	c	s	n	e	e	e	s

time analysis

SAIS

- sorts LMS recursively
- given SAIS needs $\tau(n)$ time,
  $\tau(n) = O(n) + \tau(n/2) = O(n)$ since
  the number of LMS $\leq n/2$

LSAIS

- number of LMS can be $\Theta(n)$ in all recursion steps
- still $O(n^2)$ time? 😢

example

T = b...baababaabab....aabab

b|...|b|aabab|aabab|...|aabab

example

T = b...baababaabab....aabab

b|...|b|aabab|aabab|...|aabab

LMS:

- bb → 3
- aaba → 1
- aba → 2

assign ranks according to lexicographic order

example

T = b...baababaabab....aabab

b|...|b|aabab|aabab|...|aabab

LMS:

- bb → 3
- aaba → 1
- aba → 2

assign ranks according to lexicographic order

3|....|3|12|12|...|12 (recursion step)

example

T = b...baababaabab....aabab

b|...|b|aabab|aabab|...|aabab

LMS:

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assign ranks according to lexicographic order

same amount

3|....|3|12|12|...|12 (recursion step)

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T = b...baababaabab....aabab

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LMS:

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same amount

3|....|3|12|12|...|12 (recursion step)

same Lyndon factorization

observation

- the string on the ranked LMS subtrings has the same Lyndon factorization
- compared to any character *c*, all LMS starting with *c* are larger with respect to $\prec_\omega$

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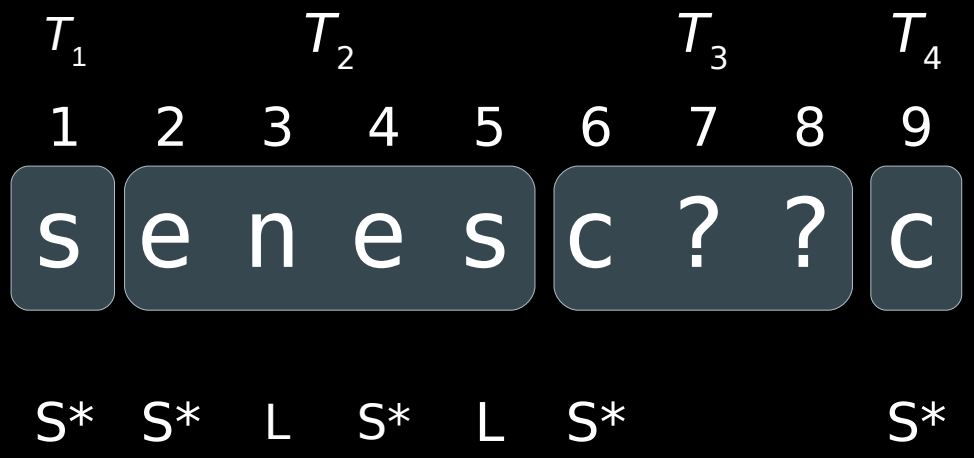

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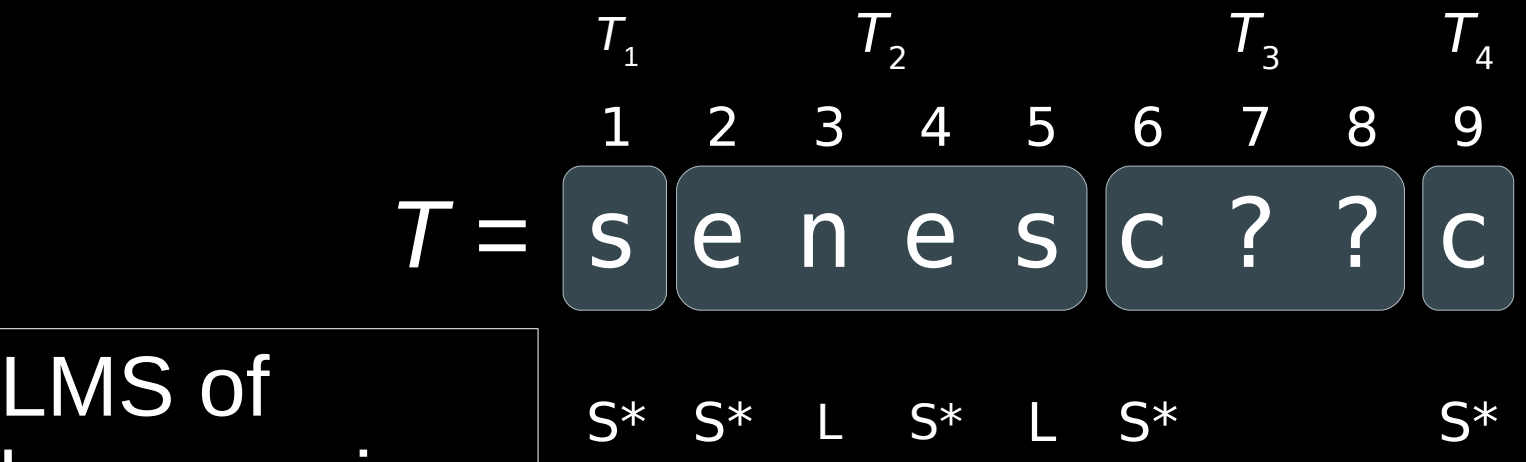

⇒ can omit LMS of length 1 in the recursion

summary

- bijective BWT construction
  - O(*n*) time on integer alphabets
- methods
  - adapt SAIS
  - sort conjugates in $\prec_\omega$ instead of suffixes in lexicographic order
  - skip LMS of length 1 in recursion

summary

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  - O(*n*) time on integer alphabets
- methods
  - adapt SAIS
  - sort conjugates in $\prec_\omega$ instead of suffixes in lexicographic order
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all questions are welcome!