Space-Efficient B Trees via Load-Balancing

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Abstract. We study succinct variants of B trees in the word RAM model that require $s + o(s)$ bits of space, where s is the number of bits essentially needed for storing keys and possibly other satellite values. Assuming that elements are sorted by keys (not necessarily in the order of their integer representations), our B trees support standard operations such as searching, insertion and deletion of elements. In some applications it is useful to associate a satellite value to each element, and to support aggregate operations such as computing the sum of values, the minimum/maximum value in a given range, or search operations based on those values. We propose a B tree representation storing n elements in $s + \mathcal{O}(s/\lg n)$ bits of space and supporting all mentioned operations in $\mathcal{O}(\lg n)$ time. Operations on integer-ordered keys and satellite values can be accelerated to $\mathcal{O}(\lg n/\lg\lg n)$ time if we use $s + \mathcal{O}(s \lg\lg n/\lg n)$ bits of space. The time is retained for special kind of aggregate functions that can be computed bit-parallel in constant time. For integer-ordered keys, we can also compress the space s to match compression measures using difference encoding of the keys while retaining the operational time complexities. For the last enhancement, we allow us to pre-compute tables of $o(n)$ bits.

Keywords: B tree, succinct data structure, predecessor data structure

1 Introduction

A B tree [2] is the most ubiquitous data structure found for relational databases and is, like the balanced binary search tree in the pointer machine model, the most basic search data structure in the external memory model. A lot of research has already been dedicated for solving various problems with B trees, and various variants of the B tree have already been proposed (cf. [15] for a survey). Here, we study a space-efficient variant of the B tree in the word RAM model under the context of a dynamic predecessor data structure, which provides the following methods:

predecessor(K) returns the predecessor of a given key K (or K itself if it is already stored);

insert(K) inserts the key K;[3] and
delete(K) deletes the key K.

We call these three operations *B tree operations* in the following. Nowadays, when speaking about B trees we actually mean B+ trees [5, Sect. 3] (also called *leaf-oriented B-tree* [3]), where the leaves store the actual data (i.e., the keys). We stick to this convention throughout the paper. Another variant we want to focus on in this paper is the B* tree [20, Sect. 6.2.4], where a node split on inserting a key into a full node has chances to be deferred by balancing the loads of this node with one of its siblings.

1.1 Related Work

The standard B tree as well its B+ and B* tree variants support the above methods in $\mathcal{O}(\lg n)$ time, while taking $\mathcal{O}(n)$ words of space for storing n keys. Even if each key uses only $k \in o(\lg n)$ bits, the space requirement remains the same since its pointer-based tree topology already needs $\mathcal{O}(n)$ pointers. To improve the space while retaining the operational time complexity in the word RAM model is the main topic of this article. However, this is not a novel idea:

The earliest approach we are aware of is due to Blandford and Blelloch [4] who proposed a representation of the leaves as blocks of size $\Theta(\lg n)$. Assuming that keys are integer of k bits, they store the keys not in their plain form, but by their differences encoded with Elias-γ code [8]. Their search tree takes $\mathcal{O}(n \lg((2^k + n)/n))$ bits while conducting B tree operations in $\mathcal{O}(\lg n)$ time.

More recently, Prezza [24] presented a B tree whose leaves store between $b/2$ and b keys for $b = \lg n$. Like [3, Sect. 3] or [7, Thm. 6], the main aim was to provide prefix-sums by augmenting each internal node of the B tree with additional information about the leaves in its subtree such as the sum of the stored values. Given m is the sum of all stored keys plus n, the provided solution uses $2n\,(\lg(m/n) + \lg\lg n + \mathcal{O}(\lg m/\lg n))$ bits of space and supports B tree operations as well as prefix-sum in $\mathcal{O}(\lg n)$ time. This space becomes $2nk + 2n\lg\lg n + o(n)$ bits if we store each key in plain k bits.

Data structures computing prefix-sums are also important for dynamic string representations [16, 21, 22]. For instance, He and Munro [16] use a B tree as underlying prefix-sum data structure for efficient deletions and insertions of characters into a dynamic string. If we omit the auxiliary data structures on top of the B tree to answer prefix-sum queries, their B tree uses $nk + \mathcal{O}(nk/\sqrt{\lg n})$ bits of space while supporting B tree operations in $\mathcal{O}(\lg n/\lg\lg n)$ time, an improvement over the $\mathcal{O}(\lg n)$ time of the data structure of González and Navarro [14, Thm. 1] sharing the same space bound. In the static case, Delpratt et al. [6] studied compression techniques for a static prefix-sum data structure.

Aside from prefix-sums, another problem is to maintain a set of strings, where each node v is augmented with the length of the longest common prefix (LCP)

[3] It is possible to support duplicate keys. The discussion is deferred to Sect. 5.3 where we study a more involving setting resembling an associated array.

among all strings stored as satellite values in the leaves of the subtree rooted at v [11].

Next, there is a line of research on implicit data structures supporting B tree operations: Under the assumption that all keys are distinct, the data structure of Franceschini and Grossi [12] supports $\mathcal{O}(\lg n)$ time for predecessor and $\mathcal{O}(\lg n)$ amortized time for updates (delete and insert) while using only constant number of words of extra space to a dynamic array A of size kn bits storing the keys. However, they assume a more powerful model of computation, where expanding or contracting A at its end can be done in constant time. This model is more powerful in the sense that the standard RAM model only supports the reallocation of a new array and copying the contents of the old array to the new array, thus taking time linear in the size of the two arrays. In the standard RAM model, arrays with such operations (extension or contraction at their ends) are called *extendible arrays*, and the best solution in this model (we are aware of) uses $nk + \mathcal{O}(w + \sqrt{knw})$ bits of space for supporting constant-time access and constant-time amortized updates [25, Lemma 1]. Allowing duplicate keys, Katajainen and Rao [19] presented a data structure with the same time bounds as [12] but using $\mathcal{O}(n \lg \lg n / \lg n)$ bits of extra space.

With respect to similar techniques but different aim, we can point out the succinct dynamic tree representation of Farzan and Munro [10, Thm 2] who propose similar techniques like rebuilding substructures after a certain amount of updates (cf. Sect. 5.1), or storing satellite data in blocks (cf. Sect. 5). They also have a need for space-efficient prefix-sum data structures.

In what follows, we present a solution for B trees based on different known techniques for succinct data structures such as [25] and the aforementioned B tree representations.

1.2 Our Contribution

Our contribution (cf. Sect. 3) is a combination of a generalization of the rearrangement strategy of the B* tree with the idea to enlarge the capacity of the leaves similarly to some approaches listed in the related work. With these techniques we obtain:

Theorem 1. *There is a B tree representation storing n keys, each of k bits, in $nk + \mathcal{O}(nk/\lg n)$ bits of space, supporting all B tree operations in $\mathcal{O}(\lg n)$ time.*

We stress that this representation does not compress the keys, which can be advantageous if keys are not simple data types but for instance pointers to complex data structure such that equality checking cannot be done by merely comparing the bit representation of the keys, but still can be performed in constant time. In this setting of incompressible keys, the space of a *succinct* data structure supporting predecessor, insert, and delete is $nk + o(nk)$ bits for storing n keys.

We present our space-efficient B tree in Sect. 3. In Sect. 4, we give some practical considerations to implement the idea of Sect. 3 and conduct experiments

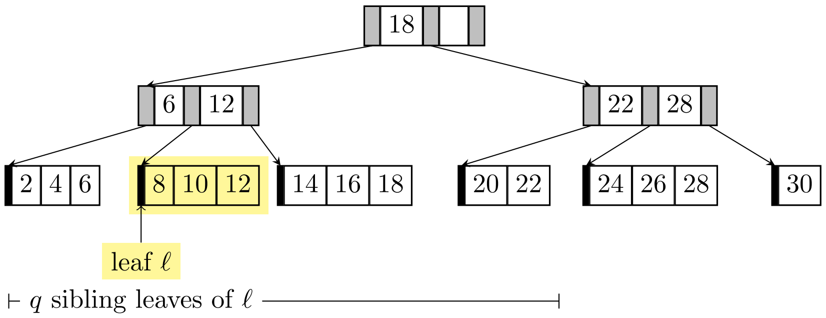


Fig. 1. A B+ tree with degree $t = 3$ and height 3. A leaf can store at most $b = t = 3$ children. A child pointer is a gray field in the internal nodes. An internal node v stores $t - 1$ integers in an array I_v where the value $I_v[i]$ regulates that only those keys of at most $I_v[i]$ go to the children in the range from the first up to the i-th child. In what follows (Fig. 2), we consider inserting the key 9 into the full leaf ℓ (storing the keys 8, 10, and 12), and propose a strategy different from splitting ℓ by considering its $q = 3$ siblings.

to evaluate them. Additionally, we show that we can augment our B tree with auxiliary data such that we can address the prefix-sum problem and LCP queries without worsening the query time (cf. Sect. 5). In Sect. 6.1, we show that we can speed up our B tree with a technique used by He and Munro [16]. Next, we show that we can compress the keys to achieve similar results as previous solutions working on compressed keys, while retaining the operational time complexities (Sect. 6.2). Note that the settings of Sect. 6.1 and Sect. 6.2 assume that we store the keys integer-ordered, i.e., based on the order of their integer representations. Finally, we show in Sect. 6.3 that our B tree variant inherits the virtues of the standard B tree in the external memory when it comes to the optimal number of page accesses for a B tree operation.

The contents in Sect. 4 and Sect. 6 are extended from the conference version of this paper which appeared in [17].

2 Preliminaries

Except for the brief excursion to the external memory model in Sect. 6.3, our computational model is the word RAM model with a word size of w bits. We assume that each key uses $k \in \mathcal{O}(w)$ bits, and that we can compare two keys in $\mathcal{O}(1)$ time. More precisely, we support the comparison to be more complex than just comparing the k-bit representation bitwise as long as it can be evaluated within constant time. Let $n \in \mathcal{O}(2^w) \cap \Omega((w \lg^2 n)/k)$ be the number of keys we store at a specific, fixed time.

A B+ tree of degree t for a constant $t \geq 3$ is a rooted tree whose nodes have an out-degree between $\lceil t/2 \rceil$ and t. See Fig. 1 for an example. All leaves are on the same height, which is $\Theta(\lg n)$ when storing n keys. The number of keys each leaf stores is between $\lfloor t/2 \rfloor$ and t (except if the root is a leaf). Each leaf is represented as an array of length t; each entry of this array has k bits. We call such an array a *leaf array*. Each leaf additionally stores a pointer to its preceding and succeeding leaf. Each internal node v stores an array of length t for the pointers to its children, and an integer array I_v of length $t-1$ to distinguish the children for guiding a top-down navigation. In more detail, $I_v[i]$ is a key-comparable integer such that all keys of at most $I_v[i]$ are stored in the subtrees rooted (a) at the i-th child u of v or (b) at u's left siblings. Since the integers of I_v are stored in ascending order (with respect to the order imposed on the keys), to know in which subtree below v a key is stored, we can perform a binary search on I_v.

A root-to-leaf navigation can be conducted in $\mathcal{O}(\lg n)$ time, since there are $\mathcal{O}(\lg n)$ nodes on the path from the root to any leaf, and selecting a child of a node can be done with a linear scan of its stored keys in $\mathcal{O}(t) = \mathcal{O}(1)$ time.

Regarding space, each leaf stores at least $t/2$ keys. So there are at most $2n/t$ leaves. Since a leaf array uses kt bits, the leaves can use up to $2nk$ bits. This is at most twice the space needed for storing all keys in a plain array. In what follows, we provide a space-efficient variant.

3 Space-Efficient B Trees

To obtain a space-efficient B tree variant, we apply two ideas. We start with the idea to share keys among several leaves (Sect. 3.1) to maintain the space of the leaves more economically. Subsequently, we can adapt this technique for leaves maintaining a *non-constant* number of keys efficiently (Sect. 3.2), leading to the final space complexity of our proposed data structure (Sect. 3.3) and Thm. 1.

3.1 Resource Management by Distributing Keys

Our first idea is to keep the leaf arrays more densely filled. For that, we generalize the idea of B* trees [20, Sect. 6.2.4]: The B* tree is a variant of the B tree (more precisely, we focus on the B+ tree variant) with the aim to defer the split of a full leaf on insertion by rearranging the keys with a dedicated sibling leaf. On inserting a key into a full leaf, we try to move a key of this leaf to its dedicated sibling. If this sibling is also full, we split both leaves up into three leaves, each having $2/3 \cdot b$ keys on average [20, Sect. 6.2.4], where $b = t$ is the maximum number of keys a leaf can store. Consequently, the number of leaves is at most $3n/2b$. We can generalize this bound by allowing a leaf to share its keys with $q \in \Theta(\lg n)$ siblings. For that, we introduce the following invariant:

Among the q siblings of every non-full leaf, there is at most one other non-full leaf.

We can leave it open to precisely specify which q siblings are assigned to which leaf. For instance, the following is possible: we can assign $q/2$ leaves to the right and to the left side of each leaf. However, if the leaf in question has $o(q)$ left siblings like the leftmost leaf, we take more of its right siblings in consideration (and by symmetry if the leaf has $o(q)$ right siblings), such that each leaf gets q siblings assigned. For this to work, we need at least q leaves, which is granted by $n \in \Omega(bq)$ as stated in Sect. 2. We note that it is possible to also accommodate smaller numbers with our techniques; we defer this analysis to Sect. 3.4.

Let us first see why this invariant helps us to improve the upper bound on the number of leaves; subsequently we show how to sustain the invariant while retaining our operational time complexity of $\mathcal{O}(\lg n)$: By definition, for every q subsequent leaves, there are at most two leaves that are non-full. Consequently, these q subsequent leaves store at least $qb - 2b$ keys. Hence, the number of leaves is at most $\lambda := nq/(qb - 2b)$, and all leaves of the tree use up to

$$\begin{aligned} \lambda bk = nqbk/(qb-2b) = nkq/(q-2) = nk + 2nk/(q-2) \\ = nk + \mathcal{O}(nk/\lg n) \text{ bits for } q \in \Theta(\lg n). \end{aligned} \tag{1}$$

To maintain the aforementioned invariant, we need to take action whenever we delete a key from a full leaf or try inserting a key into a full leaf:

Deletion When deleting a key from a full leaf ℓ having a non-full leaf ℓ' as one of its q siblings, we shift a key from ℓ' to ℓ such that ℓ is still full after the deletion. If ℓ' becomes empty, then we delete it.

Insertion Suppose that we want to insert a key into a leaf ℓ that is full. Given that one of the q sibling leaves of ℓ, say ℓ', is not full, then we shift a key from ℓ to ℓ' such that ℓ can store the new key. If there is no such ℓ', then we split ℓ. In that case, we create two new leaves, each inheriting half of the keys of the old leaf. In particular, these two leaves are the only non-full leaves among their q siblings.

It is left to analyze the time for the shifting of a key: Since each leaf stores up to $b = t \in \mathcal{O}(1)$ keys, shifting a key to one of the q siblings takes $\mathcal{O}(bq) = \mathcal{O}(\lg n)$ time. That is because, for shifting a key from the i-th leaf to the j-th leaf with $i < j$, we need to move the largest key stored in the g-th leaf to the $(g+1)$-th leaf for $g \in [i..j)$ (the moved key becomes the smallest key stored in the $(g+1)$-th leaf, cf. Fig. 2). Since a shift changes the entries of $\mathcal{O}(q)$ leaves, we have to update the information of those leaves' ancestors. By updating an ancestor node v we mean to update its integer array I_v as described in Sect. 2, which can be done in $\mathcal{O}(t)$ time. There are at most $\sum_{h=1}^{\lg n} \lceil q(t/2)^{-h} \rceil \in \mathcal{O}(\lg n + q)$ many such ancestors, and all of them can be updated in time linear to the tree height, which is $\mathcal{O}(\lg n)$ for B trees with constant degree $t \in \mathcal{O}(1)$. Thus, we obtain a B* tree variant with the same time complexities, but higher occupation rates of the leaves.

3.2 Shifting Keys Among Large Leaves

Next, we want to reduce the number of internal nodes. For that, we increase the number of elements a leaf can store up to $b := (w \lg n)/k$. Since a leaf now maintains numerous keys, shifting a key to one of its q neighboring sibling leaves

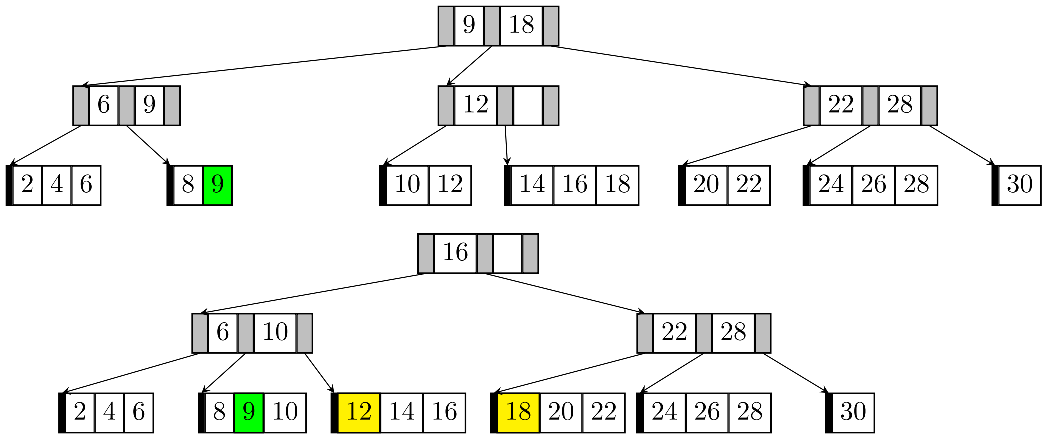


Fig. 2. Fig. 1 after inserting the key 9 into the leaf ℓ. Top: The standard B+ and B* variants split ℓ on inserting 9, causing its parent to split, too. Bottom: In our proposed variant (cf. Sect. 3) for $q \geq 3$, we shift the key 12 of ℓ to its succeeding leaf, from which we shift the key 18 to the next succeeding leaf, which was not yet full.

takes $\mathcal{O}(bqk/w) = \mathcal{O}(\lg^2 n)$ time. That is because, for an insertion into a leaf array, we need to shift the stored keys to the right to make space for the key we want to insert. We do not shift the keys individually (that would take $\mathcal{O}(b)$ total time). Instead, we can shift $\Theta(w/k)$ keys in constant time by using word-packing, yielding $\mathcal{O}(bk/w)$ time for an insertion or deletion of a key in a leaf array. In what follows, we combine the word-packing technique with *circular buffers* representing the leaf arrays to improve the time bounds to $\mathcal{O}(\lg n)$.

A *circular buffer* supports, additionally to removing or adding the last element in constant time like a standard (non-resizable) array, the same operations for the first element in constant time as well. See Fig. 4 for a visualization. For an insertion or deletion elsewhere, we still have to shift the keys to the right or to the left. This can be done in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time with word-packing as described in the previous paragraph for the plain leaf array (only extra care has to be taken when we are at the borders of the array representing the circular buffer). Finally, on inserting a key into a full leaf ℓ, we pay $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time for the insertion into this full leaf, but subsequently can shift keys among its sibling leaves in constant time per leaf. Similarly, on deleting a key of a full leaf ℓ, we first rearrange the circular buffer of ℓ in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time, and subsequently shift a key among the $\mathcal{O}(q)$ circular buffers of ℓ's siblings to keep ℓ full, which takes also $\mathcal{O}(q) = \mathcal{O}(\lg n)$ time.

3.3 Final Space Complexity

Finally, we can bound the number of internal nodes by the number of leaves λ defined in Sect. 3.1: Since the minimum out-degree of an internal node is $t/2$,

symbol	definition	where defined	set value
k	bits needed per key	Sect. 1.1	
n	number of stored keys	Sect. 1.1	
w	machine word size in bits	Sect. 2	
λ	number of total leaves	Eq. (1)	$nk + \mathcal{O}(nk/\lg n)$
t'	degree of a marginal node	Sect. 4	$\in \Omega(\lg n)$
t	degree of a B tree, i.e., number of children of an internal node	Sect. 2	$\in \mathcal{O}(1)$
		Sect. 6.1	$\lg^{\epsilon} n$
b	number of keys a leaf can store	Sect. 3.1	t
		Sect. 3.2	$w \lg n/k$
		Sect. 6.1	$w \lg n/(k \lg\lg n)$
q	number of siblings considered for key shifting	Sect. 3.1	$\in \Theta(\lg n)$
		Sect. 6.1	$\in \Theta(\lg n/\lg\lg n)$

Fig. 3. Parameters used in this paper. $\epsilon > 0$ is a user selectable parameter.

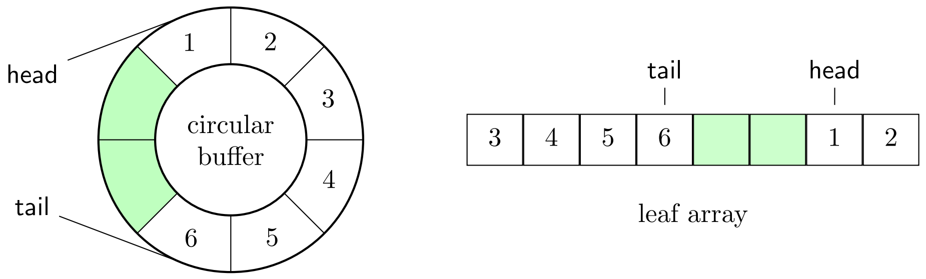


Fig. 4. A circular buffer representation of a leaf array capable of storing 8 keys. The pointers head and tail support prepending a key, removing the first key, appending a key, and removing the last key, all in constant time. The right figure shows that the circular buffer is actually implemented as a plain array with two pointers.

there are at most

$$\lambda \sum_{i=1}^{\infty} (2/t)^i = 2\lambda/(t-2) = \mathcal{O}(n(q+1)/(qtb)) = \mathcal{O}(n/(tb)) \text{ internal nodes.}$$

Since an internal node stores t pointers to its children, it uses $\mathcal{O}(tw)$ bits. In total we can store the internal nodes in

$$\mathcal{O}(twn/(tb)) = \mathcal{O}(wn/b) = \mathcal{O}(nk/\lg n) \text{ bits.} \tag{2}$$

Each circular buffer (introduced in Sect. 3.2) requires $\Theta(\lg b)$ additional bits (for the pointers in Fig. 4). The additional total space is $\lambda\,\mathcal{O}(\lg b) = \mathcal{O}(nq \lg b/(bq - 2b)) = \mathcal{O}(n \lg b/b) = o(n)$ bits. Together with Eq. (1), we finally obtain Thm. 1.

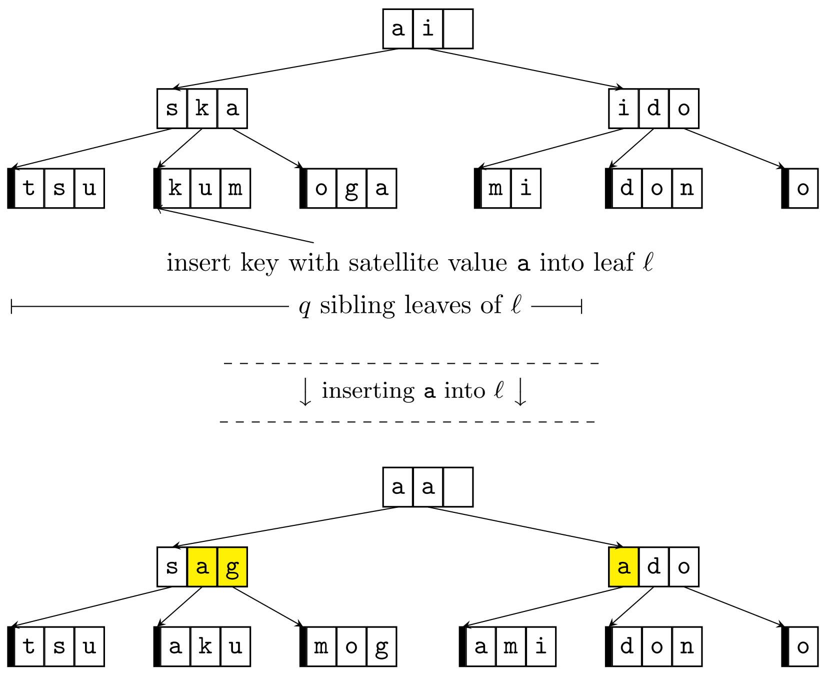


Fig. 5. Change of aggregate values on shifting keys. A shift causes the need to recompute the aggregate values of the satellite values stored in a leaf whose contents changed due to the shift. The example uses the same B tree structure as Fig. 1, but depicts the satellite values (plain characters) instead of the keys. Here, we used the minimum on the canonical Latin alphabet order as aggregate function.

3.4 Low Number of Keys

For our B tree, we require that $n \in \Omega((w \lg^2 n)/k)$. When $n \in \mathcal{O}((w \lg^2 n)/k)$ but $k \in o(n/\lg^2 n)$, we can still provide a succinct solution within the same operational time complexity, which consists of a single internal node (i.e., the root node) governing the leaves. The leaves are defined as before, except that we set the maximum number of keys a leaf can store to $b = (w \lg n)/k^2$. Consequently, the root maintains $\mathcal{O}(k \lg n)$ leaves, and for each leaf ℓ the root stores a key to delegate a search to ℓ. By maintaining this key-leaf delegation in a binary search tree, we can search and update these keys in the root in $\mathcal{O}(\lg(k \lg n)) = \mathcal{O}(\lg n)$ time. We only keep at most one non-full leaf costing us $(w \lg n)/k$ bits. The distribution of the keys among the leaves is performed as before, except that we consider, when shifting keys, all leaves instead of just the q siblings. In total, we have an overhead of $(w \lg n)/k + \mathcal{O}(k \lg^2 n) = o(n)$ bits.

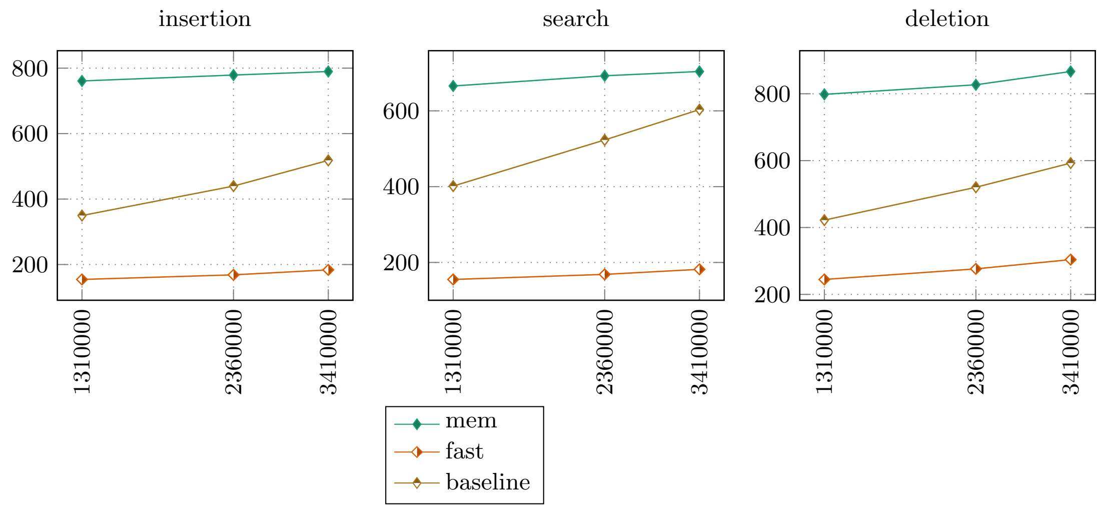


Fig. 6. Running times for two different parameter settings in comparison with the baseline `std::set<uint32_t>`. The parameter setting for *fast* is $t = 24$, $b = 96$, $q = 2$ and $t' = 48$. The parameter setting for *mem* is $t = 8$, $b = 1024$, $q = 64$ and $t' = 112$. The x-axis shows the number of elements in the set. Reported times are averages in nanoseconds per element (ns/#).

4 Practical Implementation

In what follows, we empirically evaluate our proposed B tree variation sharing elements among q sibling leaves and separating the leaf size b from the maximum out-degree t of an internal node. For that, we introduce the following simplification that introduces an additional parameter on the out-degree of the nodes having leaves as children. The motivation stems from the drawback in the theoretical proposal with respect to the number of affected parent nodes for exchanging elements among q neighboring leaves. In fact, there can be $q/t \in \mathcal{O}(\lg n)$ different parents of the leaves that need to be taken care of by an insertion or a deletion. Since we have to update the information stored in those parent nodes and their ancestors, a direct implementation is more involved than a simple recursion of updates over a single leaf-to-root path.

To restrict the updates only on leaves and a single leaf-to-root path, we enlarge the maximum out-degree of nodes having leaf children, which we call *marginal nodes* from now on. In detail, we enlarge the capacity of each marginal node to $t' \in \Omega(\lg n)$ and change the invariant of Sect. 3 to maintain the invariant only for leaves sharing the same parent.

Now only one parent of leaves is affected by an insertion or a deletion, and hence, updating the parent and above can be implemented by a simple recursion. Enlarging the capacity of marginal nodes to $\Omega(\lg n)$ ensures the theoretical invariant of having at most two non-full leaves per $\Theta(\lg n)$ leaves.

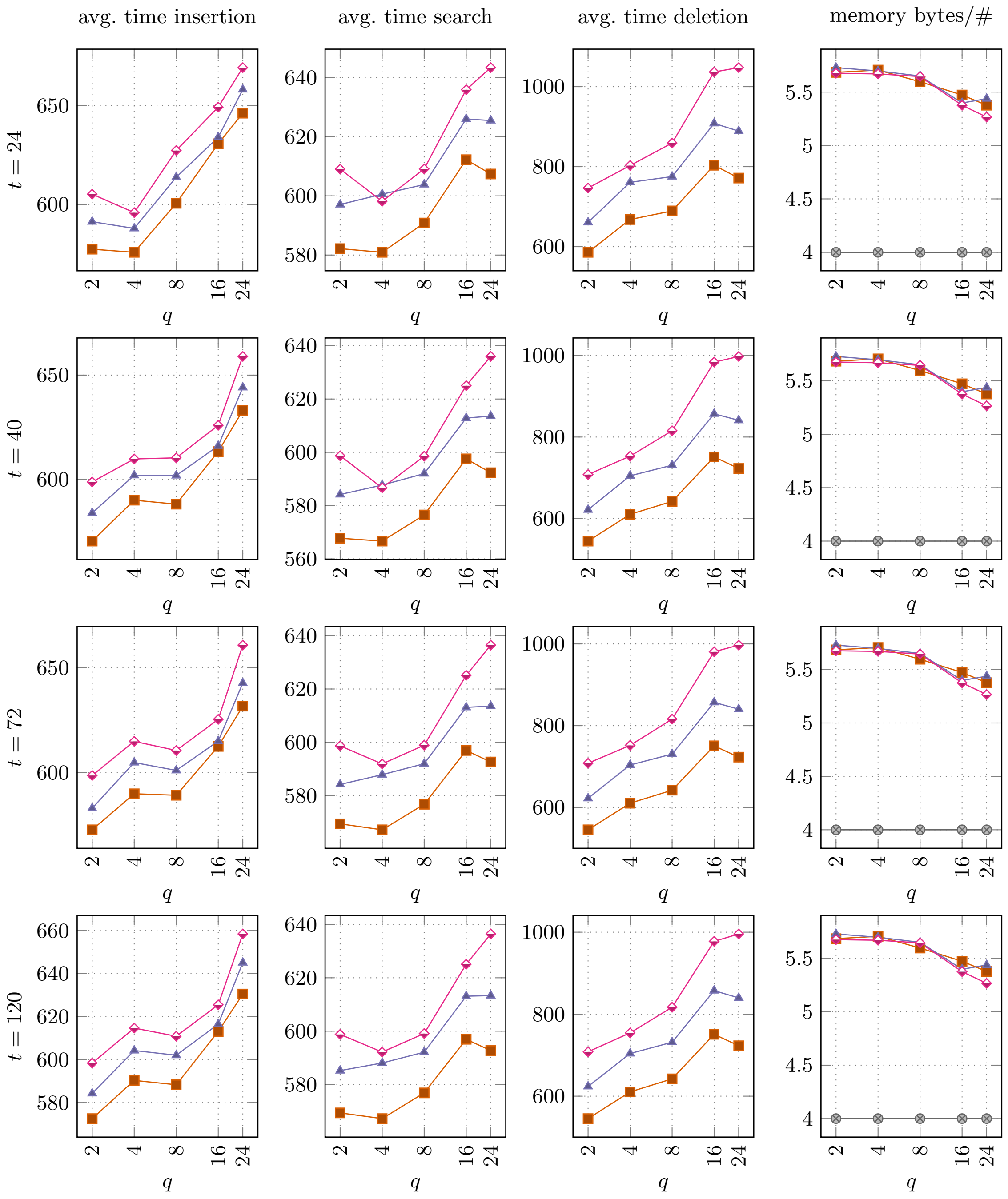
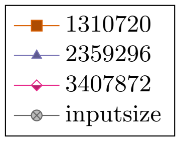


Fig. 7. Plots for $t' = 512$ and $b = 1024$ when scaling q. Reported times are averages in nanoseconds per element (ns/#).



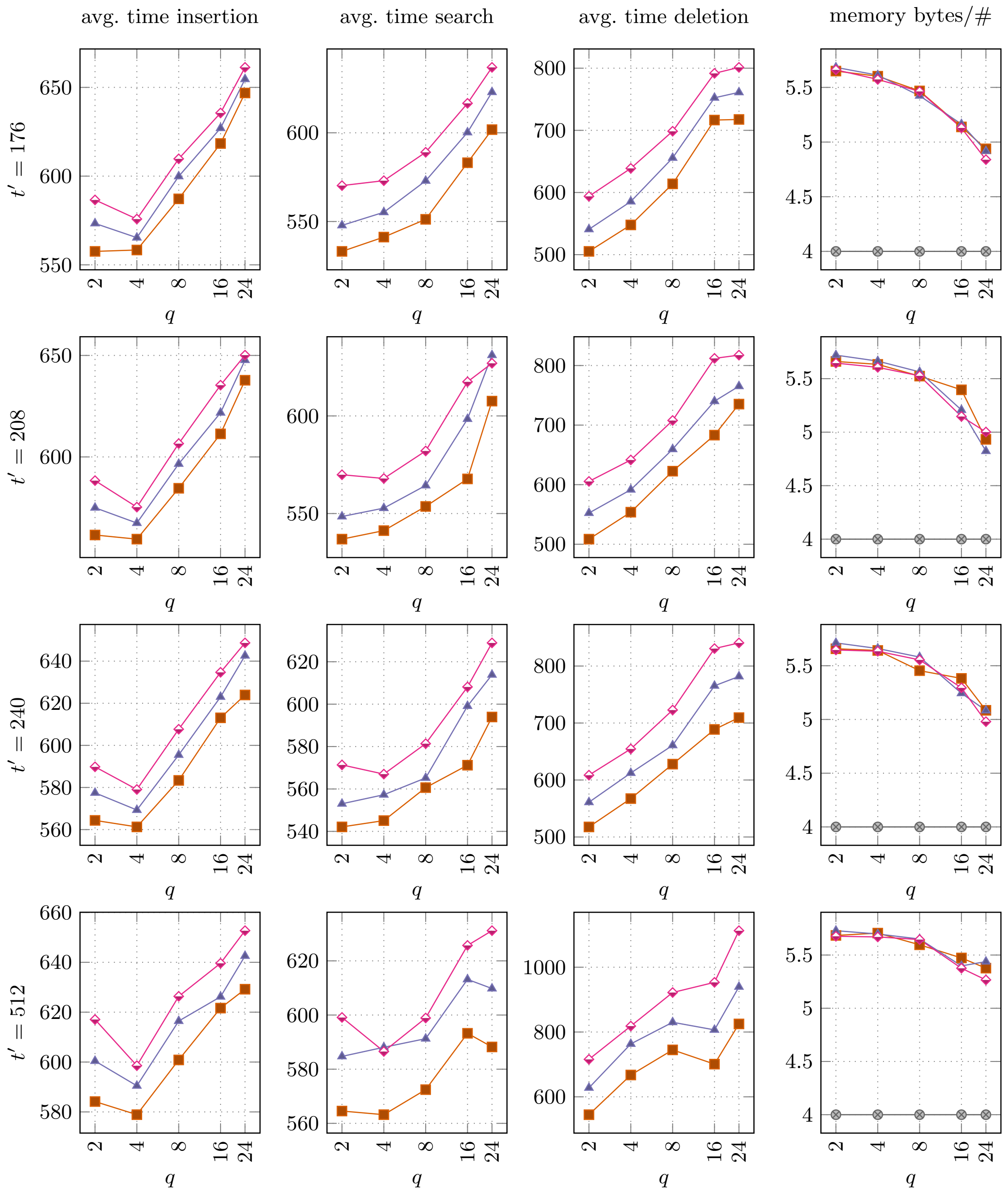
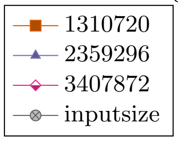


Fig. 8. Plots for $t = 8$ and $b = 1024$ when scaling q. Reported times are averages in nanoseconds per element (ns/#).



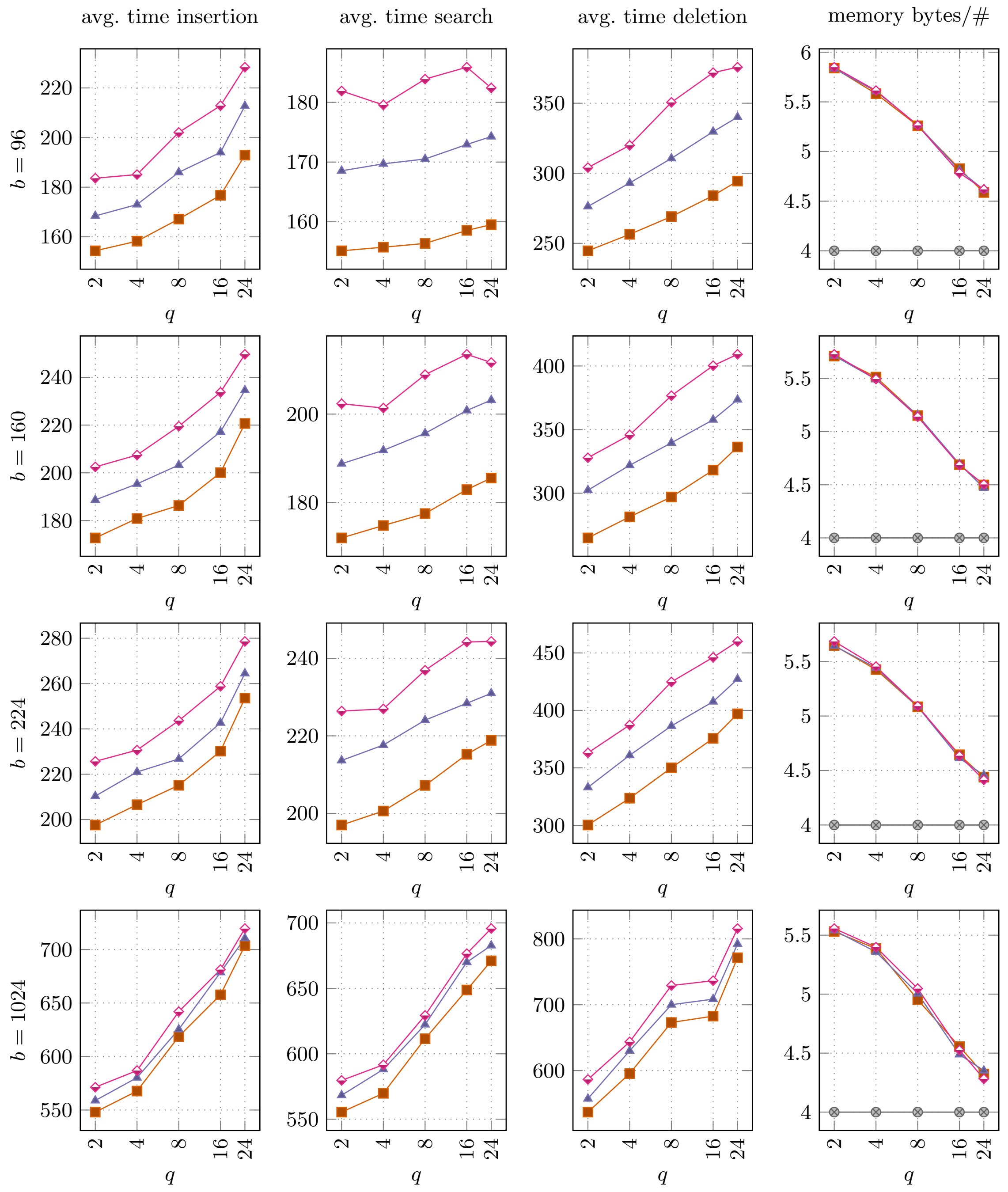
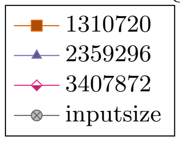


Fig. 9. Plots for $t = 24$ and $t' = 48$ when scaling q. Reported times are averages in nanoseconds per element (ns/#).



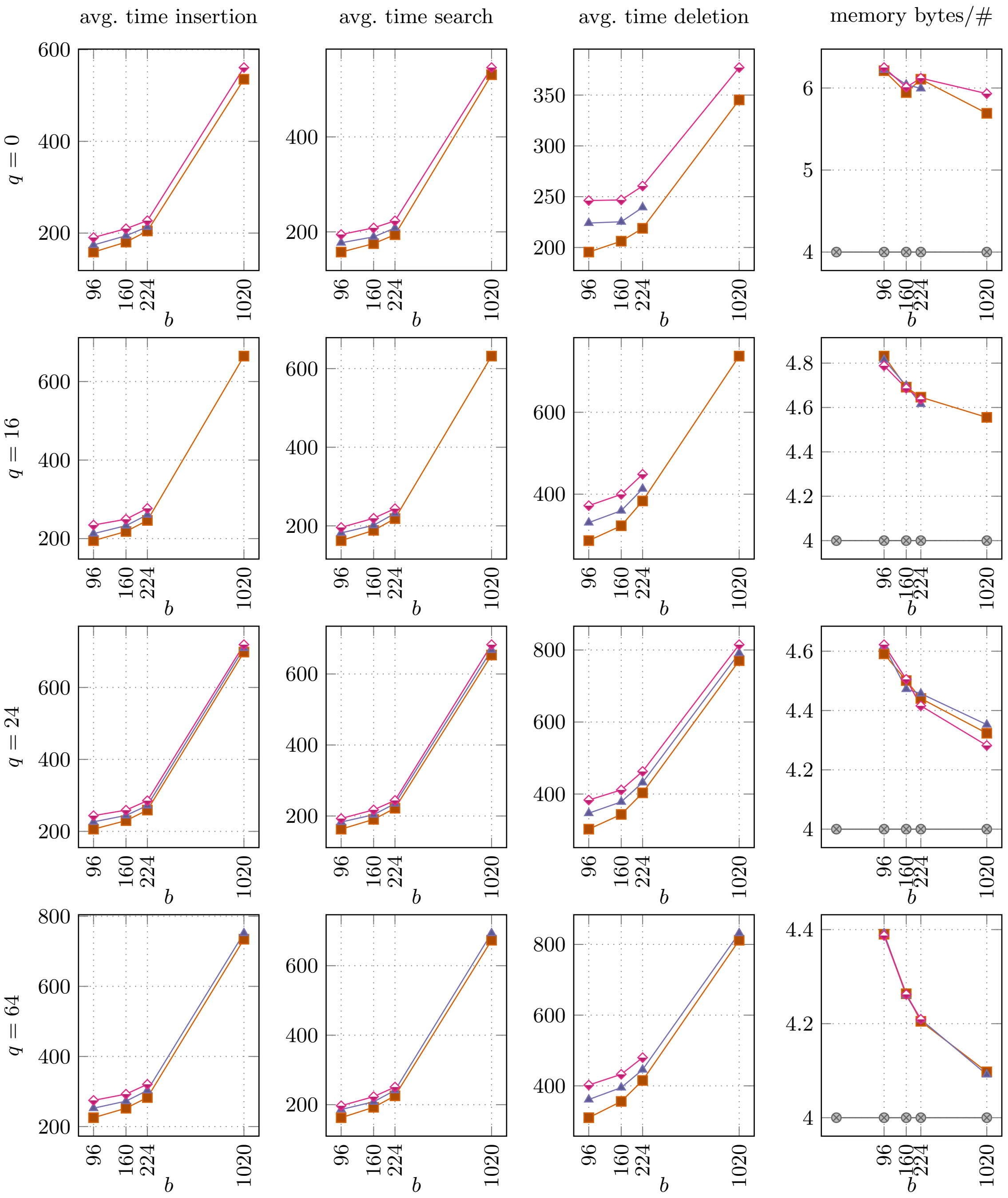


Fig. 10. Plots for $t = 8$ and $t' = 48$ when scaling b. Reported times are averages in nanoseconds per element (ns/#).

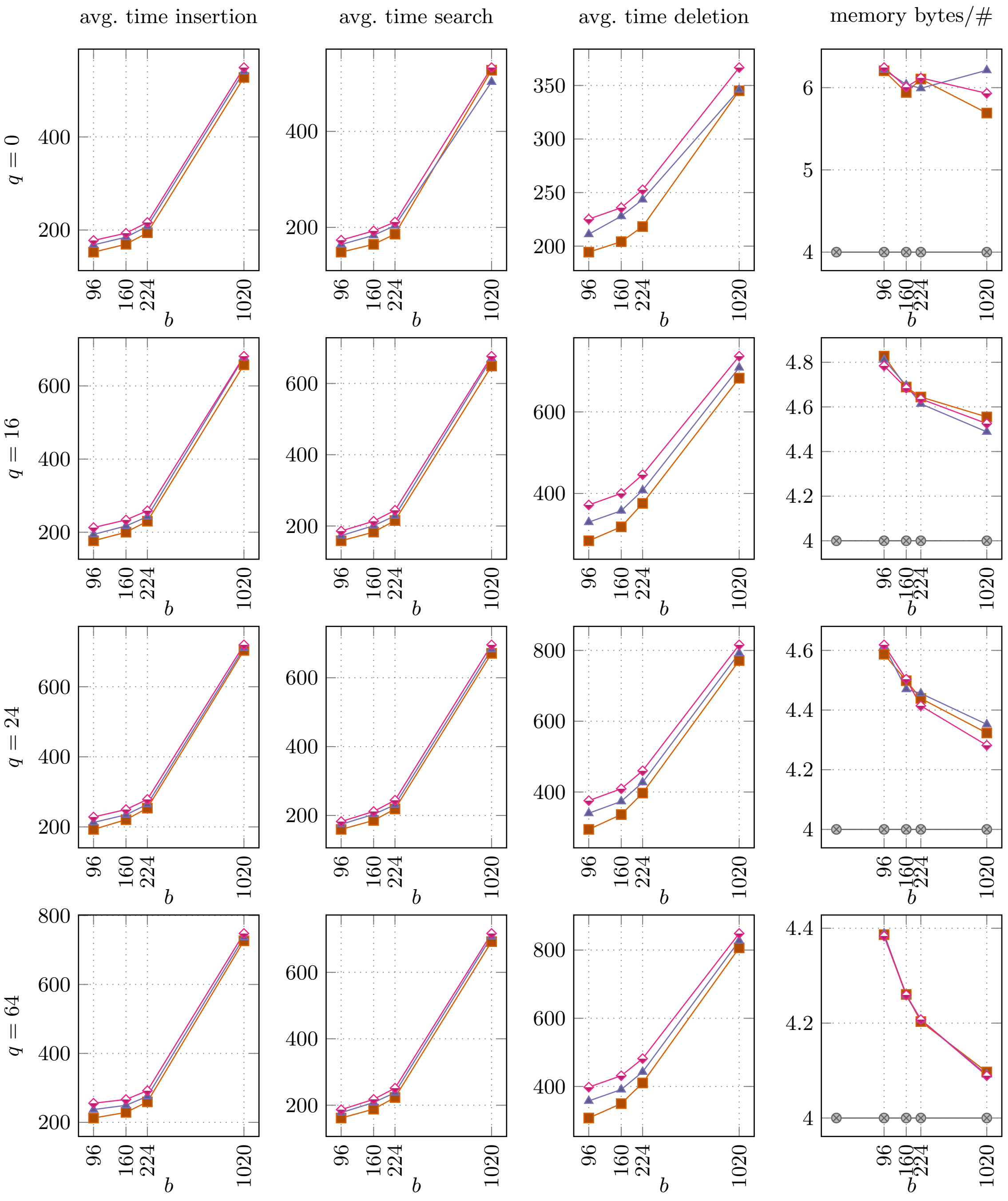
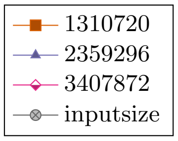


Fig. 11. Plots for $t = 24$ and $t' = 48$ when scaling b. Reported times are averages in nanoseconds per element (ns/#).



4.1 Experimental Setup

We have implemented in C++ our B tree data structure to maintain a multiset of integers. Our implementation is available at https://github.com/koeppl/btreesucc.

Our experiments were conducted on a machine with an Intel Xeon Gold 6330 CPU and Ubuntu 22.04.4 LTS. All programs were compiled by `g++ 11.4.0` compiler with optimization option `-Ofast`.

A test instance consists of the following steps:

1. allocate an array $X[1..n]$ of 32-bit integers storing sequentially the numbers from 1 to n,
2. shuffle X with the Fisher–Yates shuffle algorithm,
3. insert sequentially all elements from X into the B tree,
4. search sequentially for all elements of X in the B tree, and
5. remove sequentially all elements of X from the B tree.

We ran each experiment exactly 100 times and took average times for the insertion, search and removal operations.

4.2 Speed Evaluation

With the additional definition of t', our B tree has four parameters that impact space and time performance.

q	: number of siblings considered for key shifting,
t	: capacity of internal nodes except for marginal nodes,
t'	: capacity of marginal nodes, and
b	: number of keys a leaf can store.

While larger values of b and q trade time against potential space benefits, the parameters t and t' have rather subtle effects. For testing speed, we conducted an evaluation with two different parameter settings in Fig. 6. There, as a baseline benchmark, we compared our B tree with `std::set` of the C++ standard library in Fig. 6. Since the C++ standard does not specify an explicit implementation of `std::set`, we can empirically observe that the times and the space usage differs among different standard library implementations. On our used machine, `set<uint32_t>` takes 40 bytes per element on average[4], and thus needs much more space than our solutions we present in the following.

From Fig. 6, we also observe that the memory-friendly option with large b and q is outpaced by the baseline approach, while our speed-trimmed configuration is multiple times faster. In what follows, we study the effects of different parameter settings within both extremes.

[4] Measured with https://raw.githubusercontent.com/lemire/Code-used-on-Daniel-Lemire-s-blog/refs/heads/master/2016/09/15/stlsizeof.cpp.

4.3 Scaling q

We evaluate the effects of scaling of q with one of the other parameters. In Fig. 7, we start with scaling t while keeping t' and b fixed. There, and in the following plots, the memory bounds are accompanied by a gray baseline at 4 bytes for the input size, which is the number of bytes one of the 32-bit integers of X uses. The legend in the caption gives the used input sizes ($n \in \{1310720, 2359296, 3407872\}$). In all plots, we observe that increasing q results in higher times but better memory bounds per entry. The reason is that larger values of q help to defer the split of leaves at the expense for searching available space within q sibling leaves. The effects on scaling t seem to be more subtle. We observe, while increasing t, slightly better insertion times but slightly worse deletion times, in a majority of the cases. On the one hand, larger values of t reduce the height and thus the number of nodes visited to reach a leaf at which we perform an update. On the other hand, updating an internal node becomes more burdensome the larger t is. It thus seems that insertions take more advantage from the shorter paths to the leaves than deletions do.

Next, we scale t' in Fig. 8. While q follows the trend as in Fig. 7, it seems that smaller values of t' are favorable in time. The best space bounds are achieved for large q and $t' = 208$, which means that there is an optimal value for t' between both extremes.

Finally, we scale b in Fig. 9. The plots reassert the above assumption that larger values of b trade time against memory. Comparing $b = 1024$ with $b = 96$, the times roughly double, but the space can be improved by around 0.4 bits per element.

4.4 Two Configurations

We evaluate the configuration $t \in \{8, 24\}$ and $t' = 48$ in Figs. 10 and 11. First, we observe that scaling b has no clear effect on the memory when $q = 0$. The best times are obtained for $q = 0$ while the times slightly degenerate with larger values of q. As in the previous plots, scaling b results in larger execution times, but better memory if coupled with reasonably large values of $q > 0$.

5 Augmenting with Aggregate Values

As highlighted in the related work section (Sect. 1.1), B trees are often augmented with auxiliary data to support prefix sum queries or LCP queries when storing strings. We present a more abstract solution covering these cases with *aggregate values*, i.e., values composed of the satellite values stored along with the keys in the leaves. In detail, we augment each node v with an *aggregate value* that is the return value of a decomposable aggregate function applied on the satellite values stored in the leaves of the subtree rooted at v. A *decomposable aggregate function* [18, Sect. 2A] such as the sum, the maximum, or the minimum, is a function f on a subset of satellite values with a constant-time merge operation $\cdot_f$ such that, given two disjoint subsets X and Y of satellite values, $f(X \cup Y) = f(X) \cdot_f f(Y)$, and the left-hand and the right-hand side of the equation can be computed in the same time complexity. We further assume that each aggregate value produced by f is storable in $\mathcal{O}(w)$ bits to fit into the $\mathcal{O}(tw)$-bits space bound of an internal node.

While sustaining the methods described in the introduction like predecessor for *keys*, we enhance insert to additionally take a value as argument, and provide access to the aggregate values:

insert(K, V) inserts the key K with satellite value V;
access(v) returns the aggregate value of the node v; and
access(K) returns the satellite value of the key K.

To make use of access(v), the B tree also provides access to the root, and a top-down navigation based on the way predecessor(K) works, for a key K as search parameter. To keep things simple, we assume that all keys are distinct[5] (i.e., we allow no duplicates), although this assumption is not a necessity, as we will later see in Sect. 5.3.

For the computational analysis, let us assume that every satellite value uses $\mathcal{O}(k)$ bits, and that we can evaluate the given aggregate function f bit-parallel such that it can be evaluated in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time for a leaf storing $b = \Theta(w \lg n/k)$ values.

Under this setting, we claim that we can obtain $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time for every B tree operation while maintaining the aggregate values, even if we distribute keys among q leaves on (a) an insertion of a key into a full leaf or (b) the deletion of a key. This is nontrivial: For instance, when maintaining minima as aggregate values, if we shift the key with minimal value of a leaf ℓ to its sibling, we have to recompute the aggregate value of ℓ (cf. Fig. 5), which we need to do from scratch (since we do not store additional information about finding the next minimum value). So a shift of a key to a leaf costs $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time, resulting in $\mathcal{O}(qbk/w) = \mathcal{O}(\lg^2 n)$ overall time for an insertion.

Our idea is to decouple the satellite values from the leaf arrays where they are actually stored. To explain this idea, let us *conceptually* think of the leaf arrays as a global array — meaning that these arrays are still represented by their respective circular buffers *individually.* Given our B tree has λ leaves, we partition this global array into λ blocks, where the i-th block with $i \in [1..\lambda]$ starts initially at entry position $1+(i-1)b$, corresponds to the i-th leaf, and has initially the size equal to the capacity of the circular buffer of its corresponding leaf. The crucial change is that we let the aggregate value of a leaf depend on its corresponding block instead of its leaf array. While leaf arrays (represented by circular buffers) have a fixed capacity, we can move block boundaries freely to extend or shrink the size of a block.

Now suppose that we want to insert an element e into a full leaf ℓ, and that one of its q siblings is not full. Hence, we can redistribute one element of ℓ's leaf array by shifting one element across $\mathcal{O}(q)$ leaf arrays as explained in

[5] Note that if all keys are distinct, then $k \geq \lg n$ by the pigeonhole principle.

Sect. 3.2. After the redistribution, without the blocks, we would have to update the aggregate values of $\mathcal{O}(q)$ siblings. Instead of that, we just enlarge the block of ℓ to cover e, and update the aggregate value of ℓ with e. This allows us to process an update operation by $\mathcal{O}(q)$ block boundary updates, and a constant number of updates of the aggregate values stored in the leaves. In summary, we can decouple the aggregate values from the leaf arrays with the aid of the blocks in the global array, and therefore can use the techniques introduced in Sect. 3.2, where we shift keys among $q + 1$ sibling leaves, without the need to recompute the aggregate values of the sibling leaves when shifting keys.

Example 1. Let us assume for simplicity that $b = 3$ and that the keys are the values. Suppose that our B tree consists of exactly three leaves ℓ_i for $i = 1, 2, 3$. Each leaf ℓ_i has a leaf array A_i with the following contents: $A_1 = (1, 2, 4)$, $A_2 = (5, 6, 7)$, and $A_3 = (8, 9, \perp)$, where $\perp$ denotes an empty slot. Further assume that our aggregate function f is min such that $f(A_1) = 1, f(A_2) = 5, f(A_3) = 8$. Now suppose that we want to insert 3 into A_1. Without the block reassignment, we would shift 4 to A_2 and 7 to A_3 such that we need to update the aggregate values of ℓ_2 and ℓ_3 to $f(A_2) = 4, f(A_3) = 7$. Now, with the block reassignment, we do the following: We think of the A_i's as a single array $A[1..9] = (1, 2, 4, 5, 6, 7, 8, 9, \perp)$ and partition it initially into blocks of equal length $B_1 = A[1..3], B_2 = A[4..6], B_3 = A[7..9]$. The blocks are basically just pointers into A such that updates of A automatically update the contents of the B_i's. Now the aggregate values of the leaves are no longer based on the A_i's, but on the B_i's, i.e., $f(B_1) = 1, f(B_2) = 5, f(B_3) = 8$. If we perform the same insertion as above inserting 3 into A_1, we perform the shifting as before such that $A[1..9] = (1, 2, 3, 4, 5, 6, 7, 8, 9)$, but additionally increase the size of B_1 and shift B_2 and B_3 to the right such that $B_1 = A[1..4], B_2 = A[5..7], B_3 = A[8..9]$. Consequently, the aggregate values of ℓ_1's siblings do not have to be updated. The key observation is that while the leaf array A_1 of ℓ_1 governs $b = 3$ elements, the block B_1 of ℓ_1 is allowed to contain more/fewer than b elements.

To track the boundaries of the blocks, we augment each leaf ℓ with an offset value and the current size of its block. The offset value stores the relative offset of the block with respect to the initial starting position of the block (equal to the starting position of ℓ's leaf array) within the global array. We decrement the offset by one if we shift a key from ℓ to ℓ's preceding sibling, while we increment its offset by one if we shift a key of ℓ's preceding leaf to ℓ.

If we only care about insertions (and not about deletions and blocks becoming too large) we are done since we can update $f(X)$ to $f(X \cup \{x\})$ in constant time for a new satellite value $x \notin X$ per definition. However, deletions pose a problem for the running time because we usually cannot compute $f(X \setminus \{x\})$ from $f(X)$ with $x \in X$ in constant time. Therefore, we have to recompute the aggregate value of a block by considering all its stored satellite values. However, unlike leaf arrays whose sizes are upper bounded by b, blocks can grow beyond $\omega(b)$. Supporting deletions, we cannot ensure with our solution up so far to recompute the aggregate value of a block in $\mathcal{O}(\lg n)$ time. In what follows, we show that we

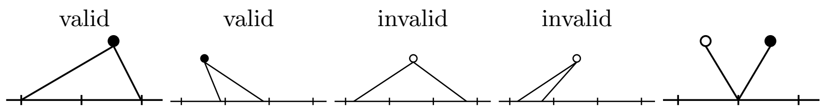


Fig. 12. Valid and invalid blocks according to the definition given in Sect. 5.2. The (conceptual) global array is symbolized by a horizontal line. The leaf arrays are intervals of the global array separated by vertical dashes. A dot symbolizes a leaf ℓ and the intersection of the triangle spawning from ℓ with the global array symbolizes the block of ℓ. A node has an invalid block if its dot is hollow. The rightmost picture shows the border case that a block is invalid if its offset is b, while a block can be valid even if it is empty.

can retain logarithmic update time, first with a simple solution taking $\mathcal{O}(\lg n)$ time amortized, and subsequently with a solution taking $\mathcal{O}(\lg n)$ time in the worst case.

5.1 Updates in Batch

Our amortized solution takes action after a node split occurs, where it adjusts the blocks of all $q + 2$ nodes that took part in that split (i.e., the full node, its q full siblings and the newly created node). The task is to evenly distribute the block sizes, reset the offsets, and recompute the aggregate values. We can do all that in $\mathcal{O}(q(bk/w + \lg n)) = \mathcal{O}(\lg^2 n)$ time, since

- there are $\mathcal{O}(q)$ leaves involved,
- each leaf stores at most b values, whose aggregate value can be computed in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n)$ time, and
- each leaf has $\mathcal{O}(\lg n)$ ancestors whose aggregate values may need to be recomputed.

Although the obtained $\mathcal{O}(\lg^2 n)$ time complexity seems costly, we have increased the total capacity of the $b + 2$ nodes involved in the update by $\Theta(b)$ keys in total. Consequently, before splitting one of those nodes again, we perform at least $b = \Omega(\lg n)$ insertions (remember that we split a node only if it and its q siblings are full). Now, whenever a block becomes larger than $2b$, we can afford the above rearrangement costing $\mathcal{O}(\lg n)$ amortized time.

5.2 Updates by Merging

To improve the time bound to $\mathcal{O}(\lg n)$ worst case time, our trick is to merge blocks and reassign the ownership of blocks to sibling leaves. For the former, a merge of two blocks means that we have to combine two aggregate values, but this can be done in constant time by the definition of the decomposable aggregate function. To keep the size of the blocks within $\mathcal{O}(b)$, we watch out for blocks whose shape underwent too much changes, which we call invalid (see Fig. 12 for a visualization). We say a block is *valid* if it covers at most $2b$ keys,

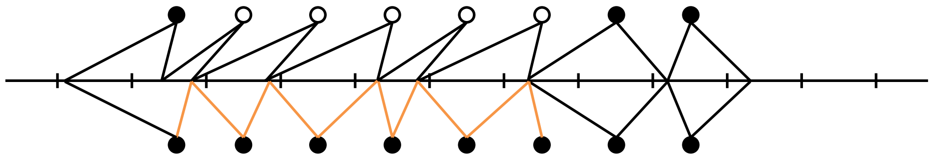


Fig. 13. Revalidation of multiple invalid blocks. The figure uses the same pictography as Fig. 12, but additionally shows on the bottom (vertically mirrored) the outcome of our algorithm fixing the invalid blocks (Sect. 5.2), where we empty the rightmost invalid block and swap the blocks until we find a block that can be merged with the previous block.

it has an offset in $(-b..b)$ (i.e., the block starts within the leaf array of the preceding leaf or of its corresponding leaf), and the sum of offset and size is in $[0..2b)$ (i.e., the block ends within the leaf array of its corresponding leaf or its succeeding leaf). Initially, all blocks are valid because they have size b and offset 0. If one of those conditions for a block becomes violated, we say that the block is *invalid*, and we take action to restore its validity. Blocks can become invalid when changing their sizes by one, or when shifting their boundaries by one. A shift can cause $\mathcal{O}(q)$ blocks to become invalid (i.e., the number of siblings considered when distributing keys). Suppose that a block B_i has become invalid due to a tree update, which already costed $\mathcal{O}(\lg n)$ time (the time for a root-to-leaf traversal). Our goal is to rectify the invalid block B_i within the same time bound. B_i has become invalid because of the events that (a) it covers $2b+1$ keys, (b) has offset $-b$ or the sum of offset and size are negative, or (c) has offset $+b$ or the sum of offset and size are at least $2b$.

For event (a), we redistribute sizes and offsets of B_i with B_{i-1} and B_{i+1}. This is possible since at least one block B_{i-1} or B_{i+1} has less than $2b$ keys. Otherwise, since they are valid blocks[6], there is no space left for B_i to have $2b+1$ keys. It is therefore possible for B_i to consign at least one key to its neighbors without making them overfull. We finish by recomputing the aggregate values of the three nodes and their ancestors, costing $\mathcal{O}(\lg n)$ total time.

The events (b) and (c) can happen when shifting blocks by one to the left (b) or to the right (c). Given B_i is the rightmost (for (b)) or the leftmost (for (c)) invalid block, we swap boundaries with the preceding blocks (for (b)) or succeeding blocks (for (c)) of B_i until finding a block whose boundaries can be extended to cover the shifted part without becoming invalid. The number of blocks we take into consideration is $\mathcal{O}(q)$, since we stop at a block B_j with $|B_j| + |B_{j+1}| \leq 2b$; if there are more blocks that do not satisfy this condition, then more than q consecutive siblings leaves are full, and a leaf split must have had occurred.

To solve (b), we proceed as follows — (c) can be solved symmetrically. First, we put B_i on a stash S (storing B_i's boundaries and its aggregate value), and

[6] More precisely, these blocks were valid at least before the enlargement of B_i, which could have triggered a shifting that invalidated either B_{i-1} or B_{i+1}.

empty B_i. Next, we check whether B_{i-1} can be extended to cover S without becoming invalid. If this is possible, we let B_{i-1} cover S, update the aggregate value of B_{i-1}, and terminate. Otherwise (B_{i-1} would become invalid), we swap B_{i-1} with S. Now B_{i-1} stores the boundaries of S. By doing so, B_{i-1} does not become invalid since the offset of B_i was $-b$ (and thus the offset of B_{i-1} becomes 0), or the sum of offset and size was in $[-b..0)$ (which becomes $[0..b)$), while the changed offset poses no problem, since the sum of offset and size of B_{i-1} is now at most $2b-1$. Finally, we iteratively select the next preceding block B_{i-2} to check whether it is mergeable with the stash S without becoming invalid (cf. Fig. 13). Since each visit of a block takes constant time (either swapping or merging contents), and we visit $\mathcal{O}(q)$ blocks, fixing all invalid blocks takes $\mathcal{O}(q) = \mathcal{O}(\lg n)$ time.

5.3 Duplicate Keys

It is possible to maintain duplicate keys with our data structure described in this section (Sect. 5) maintaining aggregate values. Therefore, we slightly change the method $\mathsf{access}(K)$ to return the list of satellite values of a key K since now a key can have multiple associated satellite values. Our time bound for answering $\mathsf{access}(K)$ becomes $\mathcal{O}(\lg n + \ell)$, where ℓ is the size of the output: Obviously, we can find the leftmost of all duplicates of K in $\mathcal{O}(\lg n)$ time as before, but then can linearly scan the leaf buckets represented by circular buffers to collect the satellite values of all duplicates.

6 Further Optimizations

In what follows, we present features that can be applied upon the techniques introduced in the previous sections. We start in Sect. 6.1 with an acceleration to $\mathcal{O}(\lg n / \lg\lg n)$ time per B tree operation by using a sophisticated dictionary for selecting a child of a node in a top-down traversal. Next (Sect. 6.2), we show that we can use our B tree in conjunction with a compression of the keys stored in each leaf. Finally, we show that our B tree variant inherits the worst case I/O complexity of the standard B tree in the external memory model when adjusting the parameters b, t, and q. For Sect. 6.1 and Sect. 6.2, we assume that the keys are stored in integer-order, i.e., in the order of their integer representation (up to now, we had not such a restriction, cf. Sect. 1.2).

6.1 Acceleration with Dynamic Arrays

When storing the keys in integer-order, we can accelerate the solutions of Sects. 3 and 5 by spending insignificantly more space in the lower term:

Theorem 2. *There is a B tree representation storing n keys, each of k bits, in $nk + \mathcal{O}(nk \lg\lg n / \lg n)$ bits of space, supporting all B tree operations in $\mathcal{O}(\lg n / \lg\lg n)$ time.*

The idea is basically the same as of He and Munro [16, Lemma 1], who used the dynamic array data structure of Raman et al. [26, Thm. 1], which is an application of the *Q-heap* of Fredman and Willard [13]. This dynamic array stores $\mathcal{O}(\lg^{\epsilon} n)$ keys in $\mathcal{O}(w \lg^{\epsilon} n)$ bits of space, and supports updates and predecessor queries, both in constant time. It can be constructed in $\mathcal{O}(\lg^{\epsilon} n)$ time, but requires a precomputed universal table of size $\mathcal{O}(n^{\epsilon'})$ bits for a constant $\epsilon' > 0$. Here, we can instead make use of the dynamic fusion tree of Patrascu and Thorup [23] that gives us the same complexities without the need of such a table. For that, we fix the degree t of the B tree to $t := \lg^{\epsilon} n$, and augment each internal node with this dynamic array to support adding or removing a child to/from a node or searching a child of a node in constant time, despite the fact that t is no longer constant. With the new degree $t \in \mathcal{O}(\lg^{\epsilon} n)$, the height of our B tree becomes $\mathcal{O}(\log_t n) = \mathcal{O}(\lg n / \lg\lg n)$ such that we can traverse from the root to a leaf node in $\mathcal{O}(\lg n / \lg\lg n)$ time. Creating or removing an internal node costs $\mathcal{O}(t)$ time, or $\mathcal{O}(1)$ time amortized since a node stores $t/2$ to t children (and hence we can charge the cost of an internal node with the creation/deletion of its $\Theta(t)$ children).

To obtain overall $\mathcal{O}(\lg n / \lg\lg n)$ time for all leaf operations, we limit the number of neighbors q to consider for node splitting or merging by setting $q := \lg n / \lg\lg n$, and the number of keys stored in a leaf by $b := w \lg n / (k \lg\lg n)$. We give an overview of all parameters used up so far in Fig. 3, which also reflects the changes done in this section. An insertion into a circular buffer maintaining b keys can therefore be conducted in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n / \lg\lg n)$ time. Overall, adjusting q and b improves the running times of the B tree operations to $\mathcal{O}(\lg n / \lg\lg n)$ time, but increases the space of the B tree: Now, the leaves need $nk + \mathcal{O}(nk \lg\lg n / \lg n)$ bits according to Eq. (1) with $q \in \mathcal{O}(\lg n / \lg\lg n)$. Also, the number of internal nodes increases, and consequently the space needed for storing the internal nodes becomes $\mathcal{O}(twn/tb) = \mathcal{O}(wn/b) = \mathcal{O}(nk \lg\lg n / \lg n)$ bits according to Eq. (2).

We can accelerate also the computation in the setting of Sect. 5 where we maintain aggregate values: There, we can compute the aggregate value of a leaf storing $b \in \Theta(w \lg n / (k \lg\lg n))$ values in $\mathcal{O}(bk/w) = \mathcal{O}(\lg n / \lg\lg n)$ time. A leaf has $\mathcal{O}(\lg n / \lg\lg n)$ ancestors, and each of them needs to be updated. Since an internal node has an out-degree of $t \in \Theta(\lg^{\epsilon} n)$, the aggregate value of an internal node is based on the $\Theta(\lg^{\epsilon} n)$ aggregate values of its children. So updating the aggregate value of an internal node costs $\mathcal{O}(\lg^{\epsilon} n)$ time since an aggregate value is storable in w bits. Finally, since the tree height is $\mathcal{O}(\lg n / \lg\lg n)$, the update with the algorithm of Sect. 5.1 costs us $\mathcal{O}(\lg^{1+\epsilon} n / \lg\lg n)$ amortized time in total. If we (a) prohibit deletions or (b) support aggregate values of size $\mathcal{O}(w / \lg^{\epsilon} n)$ bits, and require the aggregate function to be evaluable bit-parallel in constant time, then we obtain $\mathcal{O}(\lg n / \lg\lg n)$ amortized time. The solution of Sect. 5.2 can be applied as well to make the amortized time bound worst-case.

6.2 Compressing the Keys

Let us again assume that the k-bit keys are stored in the order of their integer representations. Then we can compress the keys stored in each leaf array to save space. For that, we can follow the idea of Blandford and Blelloch [4, Section 2] and Delpratt et al. [6] to store the keys by their differences in an encoded form such as Elias-γ code or Elias-δ code [9, Sect. V]. A leaf represents its $b' \in [1..b]$ keys by storing the first key in its plain form using k bits, but then each subsequent key K_i by the encoded difference $K_i - K_{i-1}$, for $i \in [2..b']$.

If our trie stores the keys $K_1, \ldots, K_n$, then the space for the differences is $\sum_{i=2}^{n}(2\lg(K_i - K_{i-1}) + 1) = \mathcal{O}(n\lg((K_n + n)/n))$ bits when using Elias-γ [4, Lemma 3.1] or Elias-δ code [6, Equation (1)]. Storing the first key of every leaf takes additional $\lambda k = nqk/(bq - 2b) = \mathcal{O}(n/\lg n)$ bits. Similar to Blandford and Blelloch [4], we can replace the nk-bits term with $\mathcal{O}(n\lg((K_n + n)/n)) = \mathcal{O}(n\lg((2^k + n)/n))$ bits with this compression in our space bounds of Thms. 1 and 2, where we implement the circular buffers as resizeable arrays. This is close to the information-theoretical lower bound for representing n keys with k bits, which is $\lg\binom{2^k}{n} = nk - n\lg n + \mathcal{O}(n) = n\lg(2^k/n) + \mathcal{O}(n)$ bits.

Since each key now uses a variable amount of bits, we have to consider how to search a key in a leaf efficiently. Under the special assumption that the word size w is $\Theta(\lg n)$, i.e., the transdichotomous word RAM model [13], a solution is to use a lookup table F for decoding all keys stored in a bit chunk [4, Lemma 2]. We query the lookup table F with a key K_i and a bit chunk of $(\lg n)/2$ bits storing the keys $K_{i+1}, \cdots$ in encoded form (the first bits representing the difference $K_{i+1} - K_i$). F outputs all keys that are stored in B fitting into $\lg n/2$ bits, plus an $\mathcal{O}(\lg\lg n)$-bits integer storing the number of bits read from B (in the case that the limited output space does not contain all keys stored in B). F can be stored in $\mathcal{O}(k\lg\lg n\lg n 2^{(\lg n)/2}) = o(n)$ bits. F can decode keys fitting into $\lg n/2$ bits in constant time. With F, we can find the (insertion) position of a key in a circular buffer in the same time complexity as in the uncompressed version. When inserting a key having a successor, we need to update the stored difference of this successor, but this can be done in constant time. We can shift keys to the left and right regarding their bit positions within a circular buffer like in the uncompressed version, since we do not need to uncompress the keys.

Theorem 3. *In the transdichotomous word RAM model with $w \in \Theta(\lg n)$, there is a B tree representation storing n integer keys, each of k bits, in $n\lg(2^k/n) + \mathcal{O}(n)$ bits of space, supporting all B tree operations in $\mathcal{O}(\lg n)$ time. The representation needs a precomputation step using $\mathcal{O}(n)$ time.*

6.3 External Memory Model

Finally, we briefly sketch that our proposed variant inherits the virtues of the B tree in the external memory (EM) model [1]. For that, let us assume that we can transfer a page of B words or Bw bits between the EM and the RAM in a constant number of I/O operations (I/Os). We assume that the RAM has a size of at least B words. We assume that Bw is much smaller than nk, otherwise we can maintain our data in a constant number of pages and need $\mathcal{O}(1)$ I/Os for all B tree operations. The standard B tree (and most of its variants) with degree $t \in \Theta(B)$ exhibits the property that every B tree operation can be processed in $\mathcal{O}(\log_B n)$ page accesses, which is worst case optimal. We can translate our techniques to the EM model as follows: First, we set the degree to $t \in \Theta(B)$ such that (a) an internal node fits into a constant number of pages, and (b) the height of our B tree is $\Theta(\log_B n)$. Consequently, a root-to-leaf traversal costs us $\mathcal{O}(\log_B n)$ I/Os. If we set the number of keys a leaf can store to $b \leftarrow (wB \log_B n)/k$, then a leaf uses $wB \log_B n$ bits, or $\log_B n$ pages. This space is maintained by a circular buffer like before, supporting insertions in $\mathcal{O}(\log_B n)$ I/Os and the insertion or deletion of the last or the first element in $\mathcal{O}(1)$ I/Os. Plugging $b = (wB \log_B n)/k$ into Eq. (1) gives $nk + \mathcal{O}(nk/\log_B n)$ bits, which we use for storing the leaves together with the keys. Consequently, the space of the internal nodes becomes $nk/(B \log_B n)$ bits (cf. Sect. 3.3). For augmenting the B tree with aggregate values as explained in Sect. 5, we assume that satellite values can be stored in $\mathcal{O}(k)$ bits, and that an aggregate value of $\mathcal{O}(1)$ pages of satellite values can be computed in $\mathcal{O}(1)$ I/Os. Unfortunately, maintaining the aggregate values naively as explained at the beginning of Sect. 5 comes with the cost of recomputing the aggregate value of a leaf, which is $\mathcal{O}(\log_B n)$ I/Os. So an insertion costs us $\mathcal{O}(\log_B^2 n)$ I/Os, including the cost for updating the $\mathcal{O}(\log_B n)$ aggregate values of the ancestors of those leaves whose aggregate values have changed. However, we can easily adapt our proposed techniques in the EM model: For the amortized analysis in Sect. 5.1, we can perform this naive update costing $\mathcal{O}(\log_B^2 n)$ I/Os after $\mathcal{O}(\log_B n)$ operations. Finally, the approach using merging (Sect. 5.2) can be translated nearly literally to the EM model. Hence, we can conclude that our space-efficient B tree variant obeys the optimal worst cast bound of $\mathcal{O}(\log_B n)$ page accesses per operation like the standard B tree, while using $nk + \mathcal{O}(nk/\log_B n)$ bits of space in EM.

7 Conclusion

We provided a space-efficient variation of the B tree that retains the time complexity of the standard B tree. It achieves succinct space when the keys are considered to be incompressible. Our main tools were the following: First, we generalized the B* tree technique to exchange keys not only with a dedicated sibling leaf but with up to q many sibling leaves. Second, we let each leaf store $\Theta(b)$ elements represented by a circular buffer such that moving a largest (resp. smallest) element of a leaf to its succeeding (resp. preceding) sibling can be performed in constant time. Additionally, we could augment each node with an aggregate value and maintain these values, either with a batch update weakening the worst case time complexities to amortized time, or with a blocking of the leaf arrays that can be maintained within the worst case time complexities.

If we store the keys in the order of their integer representations, all B tree operations can be accelerated by larger degrees t in conjunction with a smaller leaf array size b and the dynamic fusion tree of Patrascu and Thorup [23] storing the children of an internal node, resulting in smaller heights but more internal nodes, which is reflected with a small increase in the lower term space complexity. We also showed that in this setting, we can compress the keys to achieve bounds similar to other dictionaries working on compressed data.

Finally, we want to highlight that our assumptions on bit-parallel operations such as for computing the aggregate function f in Sect. 5 are of practical importance: Functions like `_mm512_min_epi32`, `_mm512_max_epi32`, or `_mm512_add_epi32` of the AVX instruction set compute the component-wise minimum, the maximum or the summation, respectively, of two arrays with 16 integers, each of 32-bits, pairwise in one instruction. (In technical terms, each of the arrays has 512 bits, and thus each stores 16 integers of 32-bits.) Assuming 32-bit satellite values, we can pack them into chucks of 512 bits, and compute an aggregated chunk of size 512 bits in bit-parallel. To obtain the final aggregate value, there are again machine instructions like `_mm512_mask_reduce_min_epi32`, `_mm512_mask_reduce_max_epi32`, or `_mm512_reduce_add_epi32` available.

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